ON A CONJECTURE OF CHABAUTY

S. P. WANG

Introduction. Let G be a connected Lie group. By a *lattice of* G, we mean a discrete subgroup Γ of G such that G/Γ has a finite invariant measure. The set of all lattices of G is denoted by $\mathcal{S}(G)$. In [1], Chabauty introduced the notion of limit of subgroups of G. A sequence $\{H_n\}$ of subgroups of G *converges to a subgroup* H if for any given compact subset K of G and neighborhood V of the identity e in G, $H\cap K\subset VH_n$ and $H_n\cap K\subset VH$ hold for sufficiently large n. Thus $\mathcal{S}(G)$ becomes a topological space with the *Chabauty topology* defined by limit of lattices. In [5], some topological properties of $\mathcal{S}(G)$ have been studied. $\mathcal{S}(G)$ is separable metric. However in general we do not know whether $\mathcal{S}(G)$ is locally compact or not. Let $A(G)$ be the group of all continuous automorphisms of G. Equipped with the compact-open topology, $A(G)$ is a Lie group. $A(G)$ operates continuously on $\mathcal{S}(G)$ with operation defined by $(\alpha, \Gamma)\to\alpha(\Gamma)$, for $\alpha\in A(G)$ and $\Gamma\in\mathcal{S}(G)$. In [1], Chabauty conjectured that for any lattice Γ of G, $A(G)\Gamma$ with induced topology from $\mathcal{S}(G)$ is homeomorphic to $A(G)/N(\Gamma)$, where $N(\Gamma)$ is the isotropy subgroup at Γ, or equivalently $A(G)\Gamma$ is locally compact. Followed by a theorem of Malcev [3], the conjecture is true for nilpotent Lie groups. For semisimple Lie groups, the author obtained some partial results in [5]. The purpose of this paper is to construct a counterexample in the case of solvable Lie groups.

1. Semidirect product of a compact group and a vector group. Let $V=\boldsymbol{R}^n$ and K a compact subgroup of $\mathrm{GL}(n, \boldsymbol{R})$. In $G=K\times V$ (in the sense of set only), we define a group structure by

$$(k, v)(k_1, v_1) = (kk_1, k_1^{-1}v + v_1)$$

for k, $k_1\in K$ and v, $v_1\in V$.

LEMMA 2. *V is the nilpotent radical of G, i.e., the maximal connected normal nilpotent subgroup of G.*

PROOF. Let $n(G)$ be the nilpotent radical of G. Clearly $n(G)\supset V$. Hence $n(G)=(n(G)\cap K)\cdot V$. Since $n(G)\cap K$ is compact, it is central in $n(G)$. However the action of K on V is faithful. It implies that $n(G)\cap K=\{e\}$. Thus $V=n(G)$.

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As V is a characteristic subgroup of G, we have then the restriction map res: $A(G)\to \mathrm{GL}(V)$ and the induced map ind: $A(G)\to A(K)$. Let $C(K)$ be the centralizer of K in $\mathrm{GL}(n, \boldsymbol{R})$ and $A(G)^0$ the identity component of $A(G)$.

LEMMA 3. *If K is commutative, then* $\mathrm{res}(A(G)^0)=C(K)^0$.

PROOF. Since K is compact and abelian, it is well known $A(K)^0=\{e\}$. Hence $\mathrm{ind}(A(G)^0)=\{e\}$. Clearly

$$\begin{aligned}(e, (\alpha k)x) &= \alpha((e, kx)) = \alpha((k, e)(e, x)(k, e)^{-1})\\ &= (k, e)\alpha((e, x))(k, e)^{-1} = (e, (k\alpha)x)\end{aligned}$$

for all $\alpha\in A(G)^0$, $k\in K$, $x\in V$. Therefore $\mathrm{res}(A(G)^0)\subset C(K)^0$. Conversely given any $\beta\in C(K)$, we define $\tilde{\beta}((k, x))=(k, \beta x)$, $(k, x)\in G$. It is obvious that $\tilde{\beta}\in A(G)$ and $\mathrm{res}(\tilde{\beta})=\beta$. Thus $\mathrm{res}(A(G)^0)=C(K)^0$.

4. **Diophantine approximation.** Let γ be an *irrational* number. As an immediate consequence of diophantine approximation, there exists an increasing sequence (a_n) of positive integers such that $(a_n\gamma-[a_n\gamma])$ converges to zero, where $[x]$ is the function of the greatest integer $\leqq x$, $x\in\boldsymbol{R}$. Denote $[a_n\gamma]$ and $a_n\gamma-[a_n\gamma]$ by b_n and c_n respectively. By an easy computation, we have

$$* \quad \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1+a_n&-a_n\\0&0&-1&1\end{pmatrix}\begin{pmatrix}1&0&\gamma&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&-b_n\\0&1&0&0\\0&0&1&a_n\\0&0&1&1+a_n\end{pmatrix} = \begin{pmatrix}1&0&\gamma&c_n\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Let Q be the subgroup of $\mathrm{GL}(4, \boldsymbol{R})$ containing all matrices of the form

$$\begin{pmatrix}\cos\theta&\sin\theta&0&0\\-\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad \theta\in\boldsymbol{R}.$$

Then the centralizer $C(Q)$ of Q in $\mathrm{GL}(4, \boldsymbol{R})$ is the subgroup of $\mathrm{GL}(4, \boldsymbol{R})$ consisting of all matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{X}} \\ 0 & 0 & & \end{pmatrix} \qquad \lambda \in \boldsymbol{R} - \{0\} \quad \text{and} \quad X \in \mathrm{GL}(2, \boldsymbol{R}).$$

Let

$$\bar{\gamma} = \begin{pmatrix} 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\gamma}_n = \begin{pmatrix} 1 & 0 & \gamma & c_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Due to our construction, $\bar{\gamma}_n \to \bar{\gamma}$ and $\bar{\gamma}_n \neq \bar{\gamma}_m$ for $n \neq m$. It is very easy to see that

$$C(Q)\gamma_n \neq C(Q)\gamma_m \qquad \text{for } n \neq m.$$

PROPOSITION 5. *$C(Q)^0\bar{\gamma}\,\mathrm{SL}(4, \boldsymbol{Z}) = \{x\bar{\gamma}y \mid x \in C(Q)^0$ and $y \in \mathrm{SL}(4, \boldsymbol{R})\}$ is not locally closed.*

PROOF. Suppose false. Then $C(Q)^0\bar{\gamma}$ is open in $C(Q)^0\bar{\gamma}\,\mathrm{SL}(4, \boldsymbol{Z})$ and $\bar{\gamma}_n \to \bar{\gamma}$. This implies $C(Q)^0\bar{\gamma}_n = C(Q)^0\bar{\gamma}$ for sufficiently large n. However this contradicts the fact that $C(Q)\bar{\gamma}_n \neq C(Q)\bar{\gamma}_m$ for $n \neq m$.

6. **A counterexample.** Let $V = \boldsymbol{R}^4$, $K = \bar{\gamma}^{-1}Q\bar{\gamma}$, $G = K \cdot V$ as constructed in §1, and $\Gamma = \boldsymbol{Z}^4 \subset \boldsymbol{R}^4$. Since G/Γ is compact, certainly Γ is a lattice of G.

MAIN THEOREM. *$A(G)\Gamma$ is not locally compact with induced topology $\mathcal{S}(G)$.*

PROOF. Suppose false. Then $A(G)^0\Gamma$, open in $A(G)\Gamma$ is locally compact. By Lemma 3, $A(G)^0\Gamma = \mathrm{res}(A(G)^0)\Gamma = C(K)^0\Gamma$. Let $\mathcal{S}_v(G)$ be the set of all lattices of G contained in V. It is clear that $\mathcal{S}_v(G)$ is a closed subset of $\mathcal{S}(G)$ and $\mathcal{S}_v(G) = \mathcal{S}(V)$. It is a well-known classical result $\mathcal{S}(V) \approx \mathrm{GL}(V)^0/N(\Gamma)$, where $N(\Gamma) = \mathrm{SL}(4, \boldsymbol{Z})$. Therefore it follows that $C(K)^0\ \mathrm{SL}(4, \boldsymbol{Z})$ is locally compact. But $C(K)^0\ \mathrm{SL}(4, \boldsymbol{Z}) = \bar{\gamma}^{-1}(C(Q)^0\bar{\gamma}\,\mathrm{SL}(4, \boldsymbol{Z}))$ is not locally compact by Proposition 5. Thus we are led to contradiction.

A REMARK. Although Chabauty's conjecture is not true in solvable Lie groups, it is still very likely that the conjecture will be valid in semisimple Lie groups supported by some indications in [**5**]. If G is semisimple with each factor of $\boldsymbol{R}$-rank≥ 2, $\mathcal{S}(G)$ is locally compact which is an easy consequence of [**7**].

References

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