# A NECESSARY CONDITION FOR PRINCIPAL CLUSTER SETS TO BE VOID

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ABSTRACT. Let $f$ be an arbitrary function from the open unit disk $D$ into the Riemann sphere $W$, and let $p$ be a point on the unit circle $C$. We prove that if the principal cluster set of $f$ at $p$ is void, then either $p$ is an ambiguous point of $f$ or the diameter of each arc-cluster set of $f$ at $p$ is greater than a fixed positive number.

An arc $\sigma \subset D$ is an *arc at $p$* if $\sigma \cup \{p\}$ is a continuous image of the closed interval $[0, 1]$. For each arc $\sigma$ at $p$, the set of all points $w \in W$ for which there exists a sequence $\{z_k\} \subset \sigma$ with $z_k \to p$ and $f(z_k) \to w$ is denoted by $C(f, p, \sigma)$, and its diameter (relative to the chordal metric) is denoted by $|C(f, p, \sigma)|$. A point $p \in C$ is an *ambiguous point of $f$* if there exist two arcs $\sigma_1$ and $\sigma_2$ at $p$ for which

$$C(f, p, \sigma_1) \cap C(f, p, \sigma_2) = \emptyset.$$

The *principal cluster set of $f$ at $p$* is the set

$$\Pi(f, p) = \bigcap C(f, p, \sigma)$$

where the intersection is taken over all arcs $\sigma$ at $p$.

THEOREM. *Let $f$ be an arbitrary function from $D$ into $W$ and let $p$ be a point on $C$. If $\Pi(f, p) = \emptyset$, then either* (1) *$p$ is an ambiguous point of $f$ or* (2) *there exists a positive number $h$ such that $|C(f, p, \sigma)| \geqq h$ for each arc $\sigma$ at $p$.*

PROOF. Suppose that condition (2) does not hold. Then, there exists a sequence $\{\sigma_k\}$ of arcs at $p$ with $|C(f, p, \sigma_k)| \to 0$. By choosing a subsequence of $\{\sigma_k\}$ if necessary, we may assume that there exists a point $w^* \in W$ for which

$$|C(f, p, \sigma_k) \cup \{w^*\}| \to 0.$$

Choose an arc $\sigma$ at $p$. If $p$ is not an ambiguous point of $f$, there exists a point

$$w_k \in C(f, p, \sigma) \cap C(f, p, \sigma_k)$$

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for each positive integer $k$. Since $C(f, p, \sigma)$ is closed and $w_k \to w^*$, it follows that $w^* \in C(f, p, \sigma)$. Then, since $\sigma$ represents an arbitrary arc at $p$, $w^* \in \Pi(f, p)$ in violation of $\Pi(f, p) = \emptyset$. Hence $p$ is an ambiguous point of $f$ and the theorem is proved.

REMARK. In case $f$ is holomorphic in $D$, this theorem is an immediate consequence of Theorem 1 in McMillan's paper [1].

## REFERENCE

1. J. E. McMillan, *Curvilinear oscillations of holomorphic functions*, Duke Math. J. **33** (1966), 495–498. MR **34** #1527.

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