ON THE EXISTENCE OF FUNDAMENTAL SOLUTIONS OF BOUNDARY PROBLEMS

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In this paper we prove an existence theorem for *fundamental solutions* (see definition below) of a large class of boundary value problems in a half space which includes the Cauchy problem for hyperbolic and parabolic operators with constant coefficients.

As a particular case, we obtain Shilov's result [4] on the existence of *Green's kernels* (see definition below) of Cauchy problems.

The technique employed is that of Fourier transform of tempered distributions. Our theorem relies on Hörmander's result on the division of a tempered distribution by a polynomial [3].

The result of this paper is related to our previous results [1] on the existence of fundamental kernels for regular elliptic boundary problems.

1. Let $P(D, D_t)$ where $D=(D_1, \cdots, D_n)$, $D_j=(1/i)(\partial/\partial x_j)$ and $D_t=(1/i)(\partial/\partial x_t)$ be a partial differential operator with constant coefficients. Let $P(\xi, \tau)$ be its characteristic polynomial, assume that the highest order coefficient of τ is independent of ξ and that all the roots of the equation in τ

$$P(\xi, \tau) = 0$$

have imaginary parts bounded below by a constant C, for all $\xi \in R^n$.

If m is the degree of $P(\xi, \tau)$ in τ, let

$$Q_1(D, D_t), \cdots, Q_m(D, D_t)$$

be m given partial differential operators with constant coefficients. The operators

$$(P(D, D_t), Q_1(D, D_t), \cdots, Q_m(D, D_t))$$

define a boundary problem in the half space

$$R_+^{n+1} = \{(x, t): x \in R^n, t > 0\}.$$

THEOREM. *Under the above conditions, there are m tempered distributions*

$$K_j(x, t) \in S'(R^n), \qquad 1 \leqq j \leqq m,$$

Received by the editors June 2, 1969.

[1] This research was supported by N.S.F. grant GP-11845.

depending upon a parameter $t>0$, *verifying the boundary problem*

$$
\begin{aligned}
&P(D, D_t)K_j(x, t) = 0 \quad \text{on} \quad R_+^{n+1} \\
&\lim_{t\to 0+} Q_l(D, D_t)K_j(x, t) = \delta_{l,j}\cdot\delta, \qquad 1 \leqq j \leqq m,
\end{aligned}
\tag{1}
$$

where $\delta_{l,j}$ *is the Kronecker symbol,* δ *the Dirac measure in* R^n *and the limit is taken in* $S'(R^n)$.

PROOF. 1. Fix ξ in R^n and let

$$\tau_1 = \tau_1(\xi), \cdots, \tau_m = \tau_m(\xi)$$

be the m roots (counting multiplicities) of

$$P(\xi, \tau) = 0.$$

Let $f_1, \cdots, f_m$ be analytic functions of a complex variable τ and define

$$R(P, f_1, \cdots, f_m) = \det f_j(\tau_l)/\prod_{k<l}(\tau_l - \tau_k).$$

It can be shown [2, pp. 231, 232] that $R(P, f_1, \cdots, f_m)$ is defined even in the case of multiple zeros, it is an analytic function of all the variables τ_l and we have the estimate

$$
|R(P, f_1, \cdots, f_m)| \leqq \prod_{j=1}^{m}\left(\sum_{k=0}^{m-1} \sup_{z\in K} \frac{|f_j^{(k)}(z)|}{j!}\right)
\tag{2}
$$

where K denotes the convex hull of the zeros $\tau_1, \cdots, \tau_m$ of P.

2. Set $f_j(\tau) = Q_j(\xi, \tau)$ and define

$$C(\xi) = R(P, Q_1, \cdots, Q_m) = \det Q_j(\xi, \tau_l(\xi))/\prod_{k<l}(\tau_l(\xi) - \tau_k(\xi)),$$
$$\xi \in R^n,$$

which is called the *characteristic function* of the given boundary problem.

$C(\xi)$ is, obviously, a symmetric function of $\tau_1, \cdots, \tau_m$, thus it can be expressed as a polynomial on the coefficients of τ in $P(\xi, \tau)$, that is to say, $C(\xi)$ is a polynomial in ξ.

3. Consider, now, the following function

$$H_j(\xi, t) = R(P, Q_1(\xi, \tau(\xi)), \cdots, e_j^{it\tau(\xi)}, \cdots, Q_m(\xi, \tau(\xi))),$$

$1 \leqq j \leqq m$, defined for all $\xi \in R^n$ and all $t \geqq 0$.

From our assumption on $P(\xi, \tau)$ it follows that there are constants A and B such that the zeros of $P(\xi, \tau)$ satisfy the inequality

$$(3) \qquad |\tau| \leqq A(|\xi|^B + 1).$$

From inequalities (2), (3) and our assumption that the imaginary parts of the roots of $P(\xi, \tau) = 0$ are bounded below by C we get for each fixed $t \geqq 0$, the estimate

$$|H_j(\xi, t)| \leqq A' |\xi|^{B'} e^{-tC}, \qquad \xi \in R^n.$$

This shows that, for each $t \geqq 0$, $H_j(\xi, t)$ defines a tempered distribution in ξ.

4. According to a result due to Hörmander [**3**], we can divide the tempered distribution $H_j(\xi, t)$ by the polynomial $C(\xi)$ and the result

$$U_j(\xi, t) = H_j(\xi, t)/C(\xi), \qquad 1 \leqq j \leqq m,$$

is a tempered distribution on R^n, depending upon $t \geqq 0$.

Next, it can be proved that each $U_j(\xi, t)$ verifies the following initial valued problem derived from (1) by taking Fourier transform on x:

$$P(\xi, D_t) U_j(\xi, t) = 0, \qquad \forall \xi \in R^n, \quad \forall t \geqq 0,$$
$$Q_l(\xi, D_t) U_j(\xi, 0) = \delta_{j,l}, \qquad l = 1, 2, \cdots, m.$$

5. Let

$$K_j(x, t) = F_\xi^{-1} U_j(\xi, t), \qquad 1 \leqq j \leqq m,$$

be the inverse Fourier transform of $U_j(\xi, t)$. For each $t \geqq 0$, $K_j(x, t) \in S'(R^n)$ and it is obvious that K_j verifies problem (1). Q.E.D.

DEFINITION. *We call* $(K_1, K_2, \cdots, K_m)$ *the fundamental solution of the boundary problem* $(P, Q_1, \cdots, Q_m)$.

A solution of the boundary value problem

$$P(D, D_t)u = 0 \quad \text{in} \quad R_+^{n+1},$$
$$Q_j(D, D_t)u|_{t=0} = g_j, \qquad 1 \leqq j \leqq m,$$

where g_j are smooth functions is given by

$$u(x, t) = \sum_{j=1}^{m} K_j(\cdot, t) * g_j,$$

the convolution being taken with respect to x.

APPLICATION. If we take as boundary operators the following ones

$$Q_1 = 1, \qquad Q_2 = D_t, \cdots, Q_m = D_t^{m-1}$$

our theorem gives, as a particular case, the *Green's kernel* of the

Cauchy problem, namely, a tempered distribution

$$G(x, t) \in S'(R^n), \qquad t \geqq 0,$$

verifying the equations:

$$\begin{aligned} P(D, D_t)G(x, t) &= 0 \quad \text{in} \quad R_+^{n+1}, \\ G(x, 0) &= 0, \\ D_tG(x, 0) &= 0, \\ \cdots\cdots&\cdots\cdots \\ D_t^{m-1}G(x, 0) &= \delta. \end{aligned}$$

References

1. J. Barros-Neto, *Kernels associated to general elliptic problems*, J. Functional Anal. **3** (1969), 173–192.

2. L. Hörmander, *On the regularity of the solutions of boundary problems*, Acta Math. **99** (1958), 225–264. MR **24**, A1503.

3. ———, *On the division of distributions by polynomials*, Ark. Math. **3** (1958), 555–568. MR **23** #A2044.

4. G. E. Šilov, *Well-posed boundary-value problems in a half-space for linear partial differential equations with constant coefficients*, Uspehi Mat. Nauk. **19** (1964), no. 3 (117), 3–52 = Russian Math. Surveys **19** (1964), no. 3, 1–52. MR **29** #4988.

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