RADICALS WITH INTEGRITY AND ROW-FINITE MATRICES

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Let $\mathfrak{S}$ be a ring, and let $\mathfrak{m}$ be any infinite cardinal. Let $\mathfrak{M}\mathfrak{S}$ be the ring of row-finite $\mathfrak{m}$-by-$\mathfrak{m}$ matrices over $\mathfrak{S}$. Patterson [**5**], [**4**] showed that $\mathfrak{J}\mathfrak{M}\mathfrak{S}$, the Jacobson radical of $\mathfrak{M}\mathfrak{S}$, is the ring $\mathfrak{M}\mathfrak{J}\mathfrak{S}$ of $\mathfrak{m}$-by-$\mathfrak{m}$ row-finite matrices over the Jacobson radical $\mathfrak{J}\mathfrak{S}$ of $\mathfrak{S}$ if and only if a right-vanishing condition due to Levitzki holds on $\mathfrak{J}\mathfrak{S}$. (See [**2**] for related material.) This condition forces $\mathfrak{J}\mathfrak{S}$ to be highly nil and is, in a sense, antithetical to integrity (no divisors of zero). For a ring $\mathfrak{S}$, let $\mathfrak{B}\mathfrak{S}$ be the ideal of row-bounded matrices in $\mathfrak{M}\mathfrak{S}$. That is, $B\in\mathfrak{B}\mathfrak{S}$ if and only if $B\in\mathfrak{M}\mathfrak{S}$, and B has all its nonzero entries (if there be any) lying exclusively in a finite subset of its columns, this subset depending upon B. It is well known [**5**], [**4**], [**6**], [**1**] that $\mathfrak{B}\mathfrak{J}\mathfrak{S}\leqq\mathfrak{J}\mathfrak{M}\mathfrak{S}\leqq\mathfrak{M}\mathfrak{J}\mathfrak{S}$. If a condition that is basically opposite to Levitzki's were to be imposed upon $\mathfrak{J}\mathfrak{S}$, we should expect a result rather far from that of Patterson's, as is indeed the case. Dr. A.D. Sands, in a private communication, has reported his obtaining the result of this paper in the special case where $\mathfrak{S}$ is the ring of p-adic integers. His method involves expressing $\mathfrak{B}\mathfrak{J}\mathfrak{S}$ as an intersection of primitive ideals.

THEOREM. *Let $\mathfrak{S}$ be a ring for which $\mathfrak{J}\mathfrak{S}$ has integrity. Then* $\mathfrak{B}\mathfrak{J}\mathfrak{S}=\mathfrak{J}\mathfrak{M}\mathfrak{S}<\mathfrak{M}\mathfrak{J}\mathfrak{S}$.

PROOF. At various stages of the proof, it will be convenient for $\mathfrak{S}$ to have a unity. Recall [**3**, Theorem 2, p. **11**] that $\mathfrak{S}$ can be embedded as an ideal in a ring $\mathfrak{S}'$ with a unity in such a way that $\mathfrak{J}\mathfrak{S}'=\mathfrak{J}\mathfrak{S}$. Now suppose that we could show that $\mathfrak{B}\mathfrak{J}\mathfrak{S}'=\mathfrak{J}\mathfrak{M}\mathfrak{S}'$. But $\mathfrak{B}\mathfrak{J}\mathfrak{S}'=\mathfrak{B}\mathfrak{J}\mathfrak{S}\leqq\mathfrak{J}\mathfrak{M}\mathfrak{S}$, so that $\mathfrak{J}\mathfrak{M}\mathfrak{S}'\leqq\mathfrak{J}\mathfrak{M}\mathfrak{S}\leqq\mathfrak{M}\mathfrak{J}\mathfrak{S}$. Since $\mathfrak{S}$ is an ideal in $\mathfrak{S}'$, $\mathfrak{M}\mathfrak{S}$ is an ideal in $\mathfrak{M}\mathfrak{S}'$, whence $\mathfrak{J}\mathfrak{M}\mathfrak{S}=\mathfrak{M}\mathfrak{S}\cap\mathfrak{J}\mathfrak{M}\mathfrak{S}'\leqq\mathfrak{J}\mathfrak{M}\mathfrak{S}'$. Thus, $\mathfrak{J}\mathfrak{M}\mathfrak{S}=\mathfrak{J}\mathfrak{M}\mathfrak{S}'$, and $\mathfrak{B}\mathfrak{J}\mathfrak{S}=\mathfrak{J}\mathfrak{M}\mathfrak{S}<\mathfrak{M}\mathfrak{J}\mathfrak{S}$. The proof is thereby reduced to the case where $\mathfrak{S}$ has a unity.

Suppose that there is some $M\in\mathfrak{J}\mathfrak{M}\mathfrak{S}\backslash\mathfrak{B}\mathfrak{J}\mathfrak{S}$. Then $M=(m(\alpha,\beta))$ where

(1) each $m(\alpha,\beta)\in\mathfrak{J}\mathfrak{S}$,

(2) α and β traverse some well-ordered index class Λ of cardinality $\mathfrak{m}$, and

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(3) for each $\alpha \in \Lambda$, only a finite number of $m(\alpha, \beta)$ are nonzero.
Since $M \notin \mathfrak{B}\mathfrak{J}\mathfrak{S}$, $M \neq 0$, the zero matrix. There must, therefore, be some row with at least one nonzero entry. In the ordering of Λ let $\alpha_1 \in \Lambda$ be the least such that the row with index α_1 has at least one nonzero entry. Let the nonzero entries occur in columns with indices $\gamma_i^1 \in \Lambda$, where $i<j$ implies that $\gamma_i^1 < \gamma_j^1$, and where $i=1, \cdots, r(1)$ for some positive integer $r(1)$. Suppose that we have found a finite increasing set of indices $\alpha_j \in \Lambda$, $\alpha_1 < \cdots < \alpha_k$, where k is a positive integer, such that α_j is least with the properties

(1) if $k \geqq 2$, $\alpha_{j-1} < \alpha_j$ for $2 \leqq j \leqq k$, where the row with index α_j has at least one nonzero entry;

(2) if the nonzero entries of the row with index α_j occur in the columns with indices $\gamma_i^j (1 \leqq i \leqq r(j))$, where $s<t$ implies that $\gamma_s^j < \gamma_t^j (1 \leqq s, t \leqq r(j))$, then, as finite sets,

$$\left\{\gamma_1^j, \cdots, \gamma_{r(j)}^j\right\} \nsubseteq \bigcup_{u=1}^{j-1} \left\{\gamma_1^u, \cdots, \gamma_{r(u)}^u\right\}$$

for $j=2, \cdots, k$ (if $k \geqq 2$).
Suppose that M has no further row with at least one nonzero entry in some column with index equal to none of the γ_i^j already used. Then $M \in \mathfrak{B}\mathfrak{J}\mathfrak{S}$, contrary to assumption. Thus, there exists least $\alpha_{k+1} \in \Lambda$ such that

(1) $\alpha_k < \alpha_{k+1}$ and the row with index α_{k+1} has at least one nonzero entry;

(2) if the nonzero entries of the row with index α_{k+1} occur in the columns with indices $\gamma_i^{k+1} (1 \leqq i \leqq r(k+1))$, where $s<t$ implies that $\gamma_s^{k+1} < \gamma_t^{k+1} (1 \leqq s, t \leqq r(k+1))$, then, as finite sets,

$$\left\{\gamma_1^{k+1}, \cdots, \gamma_{r(k+1)}^{k+1}\right\} \nsubseteq \bigcup_{u=1}^{k} \left\{\gamma_1^u, \cdots, \gamma_{r(u)}^u\right\}.$$

The induction shows that there exists a strictly ascending countable sequence of indices $\alpha_1 < \alpha_2 < \cdots$ such that

(1) the row with index α_j has at least one nonzero entry, these entries occurring in columns with indices $\gamma_i^j (1 \leqq i \leqq r(j))$, where $s<t$ implies that $\gamma_s^j < \gamma_t^j (1 \leqq s, t \leqq r(j))$;

(2) for $j \geqq 2$, α_j is least with property (1), with $\alpha_{j-1} < \alpha_j$, and with

$$\left\{\gamma_1^j, \cdots, \gamma_{r(j)}^j\right\} \nsubseteq \bigcup_{u=1}^{j-1} \left\{\gamma_1^u, \cdots, \gamma_{r(u)}^u\right\}.$$

Let $\Gamma = \bigcup_j \bigcup_{i=1}^{r(j)} \{\gamma_i^j\}$, a countable nonfinite set. Let the distinct members of Γ be enumerated by increasing size in Λ: $\delta_1 < \delta_2 < \cdots$.

Let A be an $\mathfrak{m}$-by-$\mathfrak{m}$ matrix $(a(\alpha, \beta))$, $a(\alpha, \beta) \in \mathfrak{S}$ for all $\alpha, \beta \in \Lambda$, and let the indices $1, 2, \cdots, j, \cdots$ stand for the initial members of Λ. For $j=1, 2, 3, \cdots$, let $a(j, \alpha_j)=1$ (the unity of $\mathfrak{S}$, appearing for the first time in the proof). Let all other $a(\alpha, \beta)=0$. By construction, $A \in \mathfrak{M}\mathfrak{S}$. The matrix A is used to remove all the unused rows of M, and

$$AM = \begin{pmatrix} \alpha_1\text{-st row of } M \\ \vdots \\ \alpha_j\text{-th row of } M \\ \vdots \end{pmatrix}.$$

Let D be an $\mathfrak{m}$-by-$\mathfrak{m}$ matrix $(d(\alpha, \beta))$, $d(\alpha, \beta) \in \mathfrak{S}$ for all $\alpha, \beta \in \Lambda$. For $j=1, 2, 3, \cdots$ $d(\delta_j, j)=1$, and let all other $d(\alpha, \beta)=0$. Since $\delta_1 < \delta_2 < \cdots$, $D \in \mathfrak{M}\mathfrak{S}$. The matrix D is used to remove all the columns of AM that are not used. Then $C=AMD \in \mathfrak{J}\mathfrak{M}\mathfrak{S} \backslash \mathfrak{B}\mathfrak{J}\mathfrak{S} \leqq \mathfrak{M}\mathfrak{J}\mathfrak{S}$; and $C=(c(\alpha, \beta))$, $c(\alpha, \beta) \in \mathfrak{J}\mathfrak{S}$, where there exist positive integers $s(i)$ ($i<\omega$, the first infinite ordinal) such that

(1) $c(i, s(i)) \neq 0$, while $c(i, \beta)=0$ if $\beta \in \Lambda$ and $\beta > s(i)$;

(2) the set S of the $s(i)$ is not bounded above; and

(3) if $\alpha \in \Lambda$ and if $\alpha \geqq \omega$ then $c(\alpha, \beta)=0$ for all $\beta \in \Lambda$.

The ith row ($i<\omega$) of C has the form $(c(i, 1), \cdots, c(i, s(i)), 0, 0, \cdots)$. Among all the $s(i)$ there is a least one, s_1; let i_1 be least in Λ such that $s(i_1)=s_1$. Suppose that $i_1, \cdots, i_k, s_1, \cdots, s_k \in \Lambda$, each $<\omega$, have been found such that s_j is least with respect to the requirements that $s_{j-1} < s_j = s(i)$ for at least one i ($2 \leqq j \leqq k$ if $k \geqq 2$); and i_j is least such that $s(i_j)=s_j$, $j=1, \cdots, k$. Since S is not bounded above, there exists least s_{k+1} such that $s_k < s_{k+1} =$ some $s(i)$; and let i_{k+1} be least among the i for which $s(i)=s_{k+1}$.

Let $T=(t(\alpha, \beta))$ be a matrix for which each $t(\alpha, \beta) \in \mathfrak{S}$ for all $\alpha, \beta \in \Lambda$. For $j<\omega$ let $t(j, \beta)=0$ if $\beta \neq i_j$, and let $t(j, i_j)=1$. For $\alpha \in \Lambda$, $\alpha \geqq \omega$, let $t(\alpha, \beta)=0$ for all $\beta \in \Lambda$. The matrix T rearranges the rows of $C=AMD$ so that the last nonzero entry of row i is to the left of the last nonzero entry of row j for any $j>i$. Then $T \in \mathfrak{M}\mathfrak{S}$, and $U=TC \in \mathfrak{J}\mathfrak{M}\mathfrak{S} \backslash \mathfrak{B}\mathfrak{J}\mathfrak{S} \leqq \mathfrak{M}\mathfrak{J}\mathfrak{S}$, where $U=(u(\alpha, \beta))$, $u(\alpha, \beta) \in \mathfrak{J}\mathfrak{S}$ for all $\alpha, \beta \in \Lambda$. The jth row of U ($j<\omega$) has the form $(u(j, 1), \cdots, u(j, s_j), 0, 0, \cdots)$ where $u(j, s_j) \neq 0$. Further, each $u(\alpha, \beta)=0$ for all $\beta \in \Lambda$ whenever $\alpha \in \Lambda$ and $\alpha \geqq \omega$.

Let the matrix $V=(v(\alpha, \beta))$, $v(\alpha, \beta) \in \mathfrak{S}$ for all $\alpha, \beta \in \Lambda$, have the entries $v(s_i, i)=1$ for all $i<\omega$ but where $v(\alpha, \beta)=0$ otherwise. By construction, $V \in \mathfrak{M}\mathfrak{S}$; and $W=UV \in \mathfrak{J}\mathfrak{M}\mathfrak{S} \backslash \mathfrak{B}\mathfrak{J}\mathfrak{S} \leqq \mathfrak{M}\mathfrak{J}\mathfrak{S}$ where $W=(w(\alpha, \beta))$, $w(\alpha, \beta) \in \mathfrak{J}\mathfrak{S}$ for all $\alpha, \beta \in \Lambda$, and, for all $i<\omega$, $w(i, j)$

$=u(i, s_j)$ provided $j \leqq i$, while all other $w(\alpha, \beta)=0$. The matrix V puts the last nonzero entries of $W=TAMD$ on the diagonal:

$$TAMDV = \begin{pmatrix} s(i_1) & 0 & 0 & 0 & \cdots \\ \cdot & s(i_2) & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}.$$

Let $N=(n(\alpha, \beta))$, $n(\alpha, \beta) \in \mathfrak{S}$ for all $\alpha, \beta \in \Lambda$, where $n(j, j+1)=1$ for all $j<\omega$, and where $n(\alpha, \beta)=0$ otherwise. Again, $N \in \mathfrak{M}\mathfrak{S}$, and $E=WN \in \mathfrak{J}\mathfrak{M}\mathfrak{S} \backslash \mathfrak{B}\mathfrak{J}\mathfrak{S} \leqq \mathfrak{M}\mathfrak{J}\mathfrak{S}$. Note that $E=(0 | W)$ where the 0 represents a single column of zeros.

Let $Y=(y(\alpha, \beta))$, $y(\alpha, \beta) \in \mathfrak{J}\mathfrak{S}$ for all $\alpha, \beta \in \Lambda$, be the quasi-inverse of E in $\mathfrak{J}\mathfrak{M}\mathfrak{S} \leqq \mathfrak{M}\mathfrak{J}\mathfrak{S}$. Since YE has no nonzero column with index ω or beyond, and, since $YE=Y+E$ is row-finite, the first row of Y has the form $(0, y(1, 2), \cdots, y(1, t_1), 0, 0, \cdots)$ where the $y(i, j)$ lie in $\mathfrak{J}\mathfrak{S}$, $2 \leqq t_1 < \omega$, and $y(1, t_1) \neq 0$.

Let I be the identity matrix of $\mathfrak{M}\mathfrak{S}$. Then

$$B = I - E = I - TAMDVN \in \mathfrak{M}\mathfrak{S}$$

is invertible with inverse $X=I-Y$. Note that the entry in the first row and (t_1+1)-st column of I is zero since $t_1 \geqq 2$. Now express I as XB. The first row of X is $(1, -y(1, 2), \cdots, -y(1, t_1), 0, 0, \cdots)$; the (t_1+1)-st column transpose of B is $(0, \cdots, 0, -w(t_1, t_1), *, *, \cdots)$ $=(0, \cdots, 0, -u(t_1, s_{t_1}), *, *, \cdots)$, the significant entry occurring in the t_1-st place. The inner product of these two rows is, on the one hand, $y(1, t_1)u(t_1, s_{t_1})$, and, on the other, 0. But $y(1, t_1)$ and $u(t_1, s_{t_1})$ are nonzero entries of $\mathfrak{J}\mathfrak{S}$, contradicting the integrity of $\mathfrak{J}\mathfrak{S}$ and establishing the theorem.

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