$(-1, 1)$ ALGEBRAS

IRVIN ROY HENTZEL

In a nonassociative ring A, the symbol (a, b, c) where a, b, c are elements of A is defined as $(a, b, c) = (ab)c - a(bc)$. The symbol $[a, b]$ where a, b are elements of A is defined as $[a, b] = ab - ba$. The nucleus of A, $N(A) = \{n \in A \mid (n, a, b) = (a, n, b) = (a, b, n) = 0$ for all $a, b \in A\}$. The center of A, $C(A) = \{s \in N(A) \mid [s, a] = 0$ for all $a \in A\}$. A trivial ideal of A is an ideal $I \neq 0$ of A such that $I^2 = 0$.

A $(-1, 1)$ ring A is a nonassociative ring in which the following identities are assumed to hold.

$$(a, b, c) + (a, c, b) = 0, \tag{1}$$

$$(a, b, c) + (b, c, a) + (c, a, b) = 0 \tag{2}$$

for all a, b, c elements of A. A $(-1, 1)$ algebra is a $(-1, 1)$ ring with identity which is also a finite dimensional vector space over a field F which satisfies $\alpha(ab) = a(\alpha b) = (\alpha a)b$ for all a, b elements of A, α in F. We shall not require that a subalgebra of A contain the identity of A, though we do require that it contain an identity of its own.

When hypotheses are placed on a ring as a whole, often these hypotheses imply certain properties for the center. For example, the center of a simple ring is a field. It is possible then, that hypotheses placed on the center will be reflected in the structure of the whole ring. What hypotheses are possible? Keeping in mind that the center of a simple ring is a field, one might suggest we assume that the center is simple, or semisimple. We could assume there are no ideals of the whole ring contained in the center. Or, we might assume that the center has no nilpotent elements. In this paper we show that any one of these conditions is sufficient to make a $(-1, 1)$ algebra associative. We prove the theorem:

THEOREM. *If A is a $(-1, 1)$ algebra over a field of characteristic $\neq 2, 3$, and the center of A contains no trivial A ideals, then A is associative.*

In this proof we use Albert's result that semisimple finite dimensional right alternative algebras are alternative [2], and Wedderburn's structure theorem that semisimple associative algebras are complete direct sums of matrix rings over division rings [3]. We use Maneri's result that for a $(-1, 1)$ algebra of characteristic $\neq 2, 3$, (A, A, A) is an ideal [5], and we use the results in my paper [4] in several places.

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We now start the proof of the theorem.

LEMMA 1 (ALBERT). *Let A be a finite dimensional power associative algebra. If x is a nonnilpotent element of A, then some polynomial in positive powers of x is a nonzero idempotent.*

The proof proceeds by induction on the dimension of the subspace generated by x. Or you may find the proof in [1, p. 23].

We shall call an element x invertible if there exists a y such that $xy=yx=1$, and $(a, x, y)=0$ for all elements a in the ring.

LEMMA 2. *In a $(-1, 1)$ algebra of characteristic $\neq 2$ with no idempotents $\neq 0, 1$, every element is either invertible or nilpotent.*

PROOF. In a $(-1, 1)$ ring characteristic $\neq 2$ implies $(a, x^r, x^s)=0$ for all elements a, x and for any positive integers r and s. See [4] statement (3) for a proof. This fact implies that a $(-1, 1)$ ring is power associative. By Lemma 1, every nonnilpotent element x is invertible and x^{-1} is a polynomial in x. Because x^{-1} is a polynomial in x, we have $(a, x, x^{-1})=0$.

By a Peirce decomposition of a ring A with respect to an idempotent e, we mean a decomposition of A as $A=A_{11}+A_{10}+A_{01}+A_{00}$ (additive direct sum), where $ex_{ij}=\delta_{1i}x_{ij}$, $x_{ij}e=\delta_{1j}x_{ij} \Leftrightarrow x_{ij} \in A_{ij}$, and $A_{ij}A_{mn} \subset \delta_{jm}A_{in}$. ($\delta_{ij}$ is the Kronecker delta: $\delta_{ij}=0$ if $i \neq j$, $\delta_{ii}=1$.)

THEOREM 1. *Let A be a $(-1, 1)$ ring with characteristic not 2 or 3, with an idempotent $e \neq 0, 1$, and with no trivial A ideals in the center of A. Then $A=A_{11}+A_{10}+A_{01}+A_{00}$ (Peirce decomposition) and $A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}$ is an ideal contained in the nucleus.*

PROOF. This is Theorem 2 of [4].

THEOREM 2. *If A is a $(-1, 1)$ algebra over a field F of characteristic $\neq 2, 3$ and there are no trivial A ideals in the center of A, then $A=\sum A_i+N$ (vector space direct sum) where N is contained in an ideal contained in the nucleus, A_i is a subalgebra with no idempotents except 0 and the identity of A_i, and $A_iA_j=0$ if $i \neq j$.*

PROOF. If A contains no idempotents except 0 or 1, we are through. If not, then A has an idempotent $\neq 0, 1$ and a Peirce decomposition $A=A_{11}+A_{10}+A_{01}+A_{00}$ and we can write $A=\sum_{i=0}^{1} A_i+N$ (vector space direct sum) where

(1) N is contained in an ideal in the nucleus,
(2) A_i is a nonzero subalgebra with identity e_i, and
(3) $A_iA_j=0$ if $i \neq j$.

Of all the decompositions $A = \sum_{i=1}^{n} A_i + N$ satisfying (1), (2), (3), pick one with the largest value n. I claim that in such a decomposition, each of the subalgebras A_i have no idempotents except 0 and the identity e_i of A_i. Suppose $A = \sum_{i=0}^{n} A_i + N$ satisfying (1), (2), (3) and n is maximal among all such possible decompositions. Furthermore assume some A_i has an idempotent $e \neq 0$, e_i. Without loss of generality, we may assume $i=0$. Now, by Theorem 1, $A = A_{11} + A_{10} + A_{01} + A_{00}$ (the Peirce decomposition of A with respect to e). In particular $A_0 = (A_0)_{11} + (A_0)_{10} + (A_0)_{01} + (A_0)_{00}$ and $(A_0)_{ij} \subset A_{ij}$. Thus $(A_0)_{10} + (A_0)_{01}$ is contained in an ideal contained in the nucleus. Since the sum of two ideals contained in the nucleus is again an ideal contained in the nucleus, letting $N' = N + (A_0)_{10} + (A_0)_{01}$, N' is contained in an ideal contained in the nucleus. Thus $A = (A_0)_{11} + (A_0)_{00} + \sum_{i=1}^{n} A_i + N'$ (vector space direct sum) satisfying (1), (2), (3) and this contradicts the maximality of n. We must assume that when n is maximal in such a decomposition, the A_i have no idempotents except 0 and the e_i. This finishes the proof of Theorem 2.

THEOREM 3 (ALBERT). *Every semisimple right alternative algebra A over a field of characteristic not 2 is alternative.*

This is Theorem 6 in [2]. Since $(-1, 1)$ algebras are a special class of right alternative algebras, Theorem 3 applies to $(-1, 1)$ algebras.

LEMMA 3. *Let A be a $(-1, 1)$ algebra over an algebraically closed field of characteristic $\neq 2, 3$. Suppose that every element of A is either invertible or nilpotent. Then $(A, A, A) \neq 0$ implies A contains a trivial ideal I which is contained in the center of A such that $I \subset (A, A, A)$.*

PROOF. Let R be the nil radical of A. Then $A/R \neq 0$ because A has an identity. It is clear that A/R is semisimple, so A/R is alternative. An alternative $(-1, 1)$ algebra of characteristic not 3 is associative. Thus A/I is a semisimple associative algebra. By the Wedderburn structure theorems ([**3**, p. 28, Theorem 10] and [**3**, Theorem 11, p. 32]), A/R is a complete direct sum of matrix rings over division rings. Since every element of A/R is invertible or nilpotent, A/R is a division ring. Thus A/R has no nilpotent elements. Thus R contains all the nilpotent elements of A.

If x is an element of A, let us define a linear transformation T_x on A by $aT_x = ax$. If x is an element of A, then T_x has an eigenvalue β and there exists $a \neq 0$, $a \in A$ such that $aT_x = \beta a$. Since $a(x - \beta 1) = 0$, $x - \beta 1$ is not invertible, and by assumption, must be nilpotent. Thus $x \in A$ implies $x = \beta 1 + r$ where $\beta 1$ is an element of the center of A and r is an element of the radical R of A. Thus $(A, A, A) = (R, R, R)$.

Assume $(A, A, A)\neq 0$. Since $(A, A, A)=(R, R, R)\subset R$ we have $(R, R, R)\cap R\neq 0$. Since R is nil, R is nilpotent (see [4, Theorem 4] for a proof). Thus there exists k such that $R^k=0$. The powers of R are defined inductively. $R^1=R$, $R^n=\sum_{i=1}^{n-1} R^iR^{n-i}$. Thus there exists an i such that $(R, R, R)\cap R^i\neq 0$, but $(R, R, R)\cap R^{i+1}=0$. Then $I=(R, R, R)\cap R^i$ is the ideal required for the conclusion of this lemma. To show I is an ideal, it is only necessary to show two things:

(1) that (A, A, A) is an ideal (see [5, Lemma 6]) and

(2), that powers of ideals are ideals.

To show that powers of ideals are ideals is easy and will not be done here. Given that (R, R, R) is an ideal, $RI\subset R(R, R, R)\cap RR^k\subset (R, R, R)\cap R^{k+1}=0$. $IR\subset (R, R, R)R\cap R^kR\subset (R, R, R)\cap R^{k+1}=0$. Thus $IR=RI=0$ and this suffices to show I is in the center of A.

Theorem 4. *If A is a $(-1, 1)$ algebra over an algebraically closed field of characteristic $\neq 2, 3$ and A has no trivial A ideals in the center of A, then A is associative.*

Proof. By Theorem 2, $A=\sum A_i+N$ and each A_i satisfies the hypothesis of Lemma 3. Suppose A_i is not associative. Then by Lemma 3, $(A_i, A_i, A_i)\neq 0$ implies there exists a trivial ideal I_i of A_i such that I_i is in the center of A_i and $I_i\subset (A_i, A_i, A_i)$. $I_iA_j=A_jI_i=0$ if $j\neq i$. N is contained in an ideal in the nucleus of A; therefore $(A, A, A)N=N(A, A, A)=0$. Consequently $I_iN=NI_i=0$. Thus I_i is a trivial ideal in the center of A. By hypothesis $I_i=0$. This is a contradiction. We must have $(A_i, A_i, A_i)=0$ and A_i is associative for all i. This means A is associative.

We now prove the main theorem.

Theorem 5. *If A is a $(-1, 1)$ algebra over a field of characteristic $\neq 2, 3$ and the center of A contains no trivial A ideals, then A is associative.*

Proof. Let K be the algebraic closure of F. Let $K\otimes_F A$ be the tensor product of K and A over F. Then $K\otimes_F A$ is a $(-1, 1)$ algebra over K. $K\otimes_F A$ may have nil ideals in its center. Notice that an algebra has trivial ideals in its center if and only if it has nil ideals in its center. Let I_1 be the maximal nil ideal in the center of $K\otimes_F A$. Let I_2/I_1 be the maximal nil ideal in the center of $K\otimes_F A/I_1$. Repeating this process, we have a chain of ideals $I_1\subsetneqq I_2\subsetneqq I_3 \cdots \subsetneqq I_n$ where I_{i+1}/I_i is the maximal nil ideal in the center of $K\otimes_F A/I_i$. The chain must terminate in either one of two ways. $K\otimes_F A/I_n$ has no trivial ideals in its center or $I_n=K\otimes_F A$. In the former case, by Theorem 4 we know $K\otimes_F A/I_n$ is associative. We can identify A with $1\otimes A$.

Thus we can say that in either case, $(A, A, A) \subset I_n$. However, $I_1 \cap A$ would be a trivial ideal in the center of A. Thus $I_1 \cap A = 0$ by assumption. Let us proceed by induction. If $I_r \cap A = 0$, then $I_{r+1} \cap A$ would be a trivial ideal in the center of A. Thus $I_{r+1} \cap A = 0$. Thus $I_n \cap A = 0$. Since $(A, A, A) \subset I_n$, we have $(A, A, A) = 0$ and so A is associative.

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Iowa State University