A NECESSARY CONDITION THAT TWO FINITE QUASI-FIELDS COORDINATIZE ISOMORPHIC TRANSLATION PLANES

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Given two quasi-fields it is usually easier to determine whether or not they are isotopic or anti-isotopic than it is to determine whether or not they can coordinatize isomorphic planes. The theorem given here has proved to be quite useful for this purpose. (See, for example, [2].)

A translation plane π may always be coordinatized, in the Hall sense [1], by a right quasi-field (Veblen-Weddenburn system). It is known that two quasi-fields which are either isotopic or anti-isotopic coordinatize isomorphic translation planes. This note gives a necessary condition that two finite quasi-fields which are neither isotopic nor anti-isotopic can coordinatize isomorphic translation planes.

For the remainder of this note let F_1, F_2 be finite quasi-fields and π_1, π_2 the associated translation planes in the sense of Hall [1]. For $i=1, 2$ let $F_{i\rho}=\{a\in F_i \mid a\neq 0 \text{ and } (xy)a=x(ya)\ \forall x,\ y\in F_i\}$ = right nucleus of F_i and let $F_{i\mu}=\{a\in F_i \mid a\neq 0 \text{ and } (xa)y=x(ay)\}$ = middle nucleus of F_i. It is easy to see that for each $a\in F_{i\rho}$ the mapping β_a: $(x, y)\rightarrow(x, ya)$, $(m)\rightarrow(ma)$, $Y_i=(\infty)\rightarrow Y_i$ is a perspectivity of π_i with center Y_i and axis the line $y=0$. The correspondence $a\leftrightarrow\beta_a$ is an isomorphism between $F_{i\rho}$ and the set of $Y_i-O_iX_i$ perspectivities of π_i. Also, for each $a\in F_{i\mu}$, the mapping γ_a: $(x, y)\rightarrow(xa, y)$, $(m)\rightarrow(mL_{(a)}^{-1})$, $Y_i\rightarrow Y_i$ is a $X_i-O_iY_i$ perspectivity of π_i and the correspondence $a\leftrightarrow\gamma_a$ is an isomorphism between $F_{i\mu}$ and the set of $X_i-O_iY_i$ perspectivities of π_i.

We will use the conventional notation $Y=(\infty)$, $O=(0, 0)$, $X=(0)$. We will also use the symbol $|A|$ to denote the cardinality of a set A.

THEOREM. *Suppose F_1 and F_2 are neither isotopic nor anti-isotopic and that π_1 and π_2 are isomorphic. Let σ be an isomorphism from π_1 to π_2 with $O_1\sigma=O_2$, $X_1\sigma=X_3$, $Y_1\sigma=Y_3$. Let $D=\{X_2,\ Y_2\}\cap\{X_3,\ Y_3\}$. Assume that if there is an isomorphism from π_1 to π_2 such that $D\neq\emptyset$ then σ is such an isomorphism. Then*

(1) *If $|D|=1$ then $(|F_{1\rho}|,\ |F_{2\rho}|)=(|F_{1\mu}|,\ |F_{2\mu}|)=1$.*
(2) *If $|D|=0$ then $(|F_{1\rho}|,\ |F_{2\rho}|)$ and $(|F_{1\mu}|,\ |F_{2\mu}|)\leqq 2$.*

Received by the editors May 12, 1969.

PROOF. Since F_1 and F_2 are neither isotopic nor anti-isotopic $|D|<2$. We will prove the theorem for the right nuclei. The proof for the middle nuclei is the same except for the appropriate word replacements. Let A be the set of all $Y_2-O_2X_2$ perspectivities of π_2 and let B be the set of all $Y_3-O_2X_3$ perspectivities of π_2. Then $|A| = |F_{2\rho}|$ and $|B| = |F_{1\rho}|$. Let $M=\{p\in L_{2\infty} \mid p=X_2\alpha$ or $p=Y_2\alpha$ for some collineation α of $\pi_2\}$. Then A is a group of permutations on $M-\{X_2, Y_2\}$ such that each orbit contains $|F_{2\rho}|$ elements. Thus $|M| =k|F_{2\rho}|+2$ for some integer k. Also, B is a group of permutations on $M-\{X_3, Y_3\}$ with each orbit containing $|F_{1\rho}|$ elements so that $|M| =t|F_{1\rho}|+|D|$ and so $(|F_{1\rho}|, |F_{2\rho}|)\leqq 2$. If $|D|=1$ then $(|F_{1\rho}|, |F_{2\rho}|)=1$.

COROLLARY. *In the theorem, if $D=\varnothing$ then the points of L_∞ in π_2 or π_1) may be partitioned into three mutually disjoint sets A_1, A_2, A_3 such that*

$$|A_i| \equiv 0 \bmod |F_{j\rho}| \text{ for } i, j=1, 2, \text{ and } i\neq j,$$
$$|A_i| \equiv 2 \bmod |F_{i\rho}| \text{ for } i=1, 2 \text{ and}$$
$$|A_3| \equiv 0 \bmod (LCM[|F_{1\rho}|, |F_{2\rho}|]).$$

PROOF. Since $D=\varnothing$, $|D| =0$ so that $|M| \equiv 2 \bmod |F_{2\rho}|$ and $|M| \equiv 0 \bmod |F_{1\rho}|$. Let $M=A_2$. Let $A_1=\{p\in L_{2\infty} \mid p=X_3\alpha$ or $p=Y_3\alpha$ for some collineation α of $\pi_2\}$. Clearly the proof of the theorem with the appropriate change of symbols shows that $|A_1| \equiv 0 \bmod |F_{2\rho}|$ and $|A_1| \equiv 2 \bmod |F_{1\rho}|$. $D=\varnothing$ implies $A_1\cap A_2=\varnothing$.

Let $A_3=L_{2\infty}-(A_1\cup A_2)$. Then each of A and B acts as a permutation group on A_3 with orbits of size $|F_{2\rho}|$ and $|F_{1\rho}|$ respectively. Thus, $|A_3| \equiv 0 \bmod(LCM[|F_{1\rho}|, |F_{2\rho}|])$.

REFERENCES

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