H-MANIFOLDS HAVE NO NONTRIVIAL IDEMPOTENTS[1]

ROBERT F. BROWN

For a set X and a function $m: X \times X \to X$, call $x \in X$ an *idempotent* (*element*) of (X, m) if $m(x, x) = x$. If (X, m) is a group with unit element e, then of course e, the *trivial idempotent*, is the only one in (X, m). At the other extreme, if $m: X \times X \to X$ is defined by $m(x, x') = x$, then (X, m) is a semigroup in which every element is idempotent. We remark that the set of idempotents plays an important part in both the algebraic and topological theories of semigroups.

A triple (X, m, e) is an *H-space* if X is a topological space, $e \in X$, and $m: X \times X \to X$ is a map such that $m(x, e) = m(e, x) = x$ for all $x \in X$. This generalization of the group concept is certainly of as much interest in topology as that of semigroup. One is led, therefore, to ask whether an H-space is like a topological semigroup, with its rich theory of idempotents, or like a topological group, which has no nontrivial idempotents at all. The title of the paper gives the answer that, at least when the underlying space is a manifold, an H-space behaves much like a topological group in this respect.

One must, however, be careful in interpreting the statement above. For example, the interval $I = [0, 1]$ is a manifold and $(I, m, 0)$ is an H-space when $m: I \times I \to I$ is defined by $m(s, t) = st + |s - t|$ and yet $(I, m, 0)$ has a nontrivial idempotent. Observe that if we define, for $0 \leqq r \leqq 1$,

$$m_r(s, t) = \frac{st}{1 + r} + |s - t|$$

then $(I, m_r, 0)$ is an H-space. We thus have a "path" of H-spaces from the H-space $(I, m, 0)$ with a nontrivial idempotent to one without such an idempotent. We claim that this is the general situation. The point is that, unlike a group product, an H-space product is not thought of as a single function, but rather as an equivalence class of functions. With this interpretation, the statement in the title is correct.

Let X be a space, $e \in X$, and let $H_e(X)$ denote the (possibly empty) space of maps $m: X \times X \to X$ such that (X, m, e) is an H-space. Two elements of $H_e(X)$ are considered to be homotopic if they are in the same path component.

Received by the editors April 23, 1969.

[1] The preparation of this paper was sponsored in part by NSF Grant GP-6530.

THEOREM. *Let (M, m, e) be an H-space where M is a compact connected triangulable n-dimensional manifold with or without boundary, $n \geqq 3$. There exists $m' \in H_e(M)$ homotopic to m such that (M, m', e) has no nontrivial idempotents.*

The theorem contains new information even about Lie groups. For example, of course the two path components of $H_e(S^3)$ which contain group products contain maps without nontrivial idempotents, but now we know also that the other ten path components of $H_e(S^3)$ have the same property.

The proof of the theorem is an application of fixed point theory. Let $f: M \rightarrow M$ be a map such that $f(e) = e$. Fixed points x and x' of f are *equivalent* if there is a path $C: I \rightarrow M$ from x to x' such that C and fC are homotopic by a homotopy which keeps x and x' fixed. Write the abelian group $\pi_1(M, e)$ in additive notation and define $F_*: \pi_1(M, e) \rightarrow \pi_1(M, e)$ by $F_*(\alpha) = f_*(\alpha) - \alpha$, where f_* is the endomorphism of $\pi_1(M, e)$ induced by f. The number of equivalence classes of fixed points of f is less than or equal to the order of the cokernel of F_* [**3**, Theorem 3.1].

Define $s: M \rightarrow M$ by $s(x) = m(x, x)$, then $s_*(\alpha) = 2\alpha$ for all $\alpha \in \pi_1(M, e)$ so S_* is the identity function and consequently all the fixed points of s are equivalent. Therefore, by Wecken's Theorem [**5**], there is a map $s': M \rightarrow M$ homotopic to s such that s' has only a single fixed point. We will next examine a proof of Wecken's Theorem in order to show that we can choose e to be the single fixed point of s' and that there is a homotopy $h_t: M \rightarrow M$ from s to s' such that $h_t(e) = e$ for all $t \in I$.

Given $\epsilon > 0$, there is a map $\bar{s}: M \rightarrow M$ with only a finite number of fixed points and a homotopy $s_t: M \rightarrow M$ such that $s_0 = s$, $s_1 = \bar{s}$, and for d the metric of M, $d(s(x), s_t(x)) < \epsilon$ for all $x \in M$ and $t \in I$ [**2**]. Cover M with a finite number of euclidean neighborhoods and take a triangulation of M of mesh less than half the Lebesgue number of the cover. Choose the ϵ above small enough so that e and $\bar{s}(e)$ lie in simplices of M that have a common face. By Lemma 1.2 of [**4**], there is a homotopy $\bar{s}_t: M \rightarrow M$ such that $\bar{s}_0 = \bar{s}$, the path $\{\bar{s}_t(e) \mid t \in I\}$ is contained in the union of the closed simplices of M containing e and $\bar{s}(e)$, and the fixed points of $\bar{s}_1$ consist of e together with the fixed points of $\bar{s}$.

Observe that s and $\bar{s}_1$ are homotopic by means of s_t followed by $\bar{s}_t$. Furthermore, the loop L_e consisting of the path $\{s_t(e) \mid t \in I\}$ followed by the path $\{\bar{s}_t(e) \mid t \in I\}$ is contained in an euclidean neighborhood and thus L_e represents the unit element of $\pi_1(M, e)$. Consequently, by the homotopy extension theorem, there is a homotopy $\bar{h}_t: M \to M$ such that $\bar{h}_0 = s$, $\bar{h}_1 = \bar{s}_1$, and $\bar{h}_t(e) = e$ for all $t \in I$. The map $\bar{s}_1$ induces the same endomorphism of $\pi_1(M, e)$ as s does, so all the fixed points of $\bar{s}_1$ are equivalent.

The proof of Theorem 2.4 of [4] tells us that there is a map $s^*: M \to M$, homotopic to $\bar{s}_1$ by a homotopy keeping e fixed, such that the only fixed points of s^* are e and another point x_0 in the same simplex of M. Of course e and x_0 are equivalent fixed points of s^*. Furthermore, an argument within that same proof assures us that there is no loss of generality in assuming that the line segment from e to x_0 and its image under s^* are homotopic by a homotopy keeping e and x_0 fixed. Therefore, we can apply Lemma 2.2 of [4] to obtain a map $s': M \to M$ whose only fixed point is e, and a homotopy h_t' from s^* to s'. An examination of the proof of Lemma 2.2 shows that $h_t'(e) = e$ for all $t \in I$. We have proved that, given $s: M \to M$ defined by $s(x) = m(x, x)$, there is indeed a map $s': M \to M$ with a single fixed point at e and a homotopy $h_t: M \to M$ from s to s' such that $h_t(e) = e$ for all $t \in I$.

Let $\Delta(M) = \{(x, x) \in M \times M\}$ and set

$$A = (M \times \{e\}) \cup (\{e\} \times M) \cup \Delta(M) \subset M \times M.$$

For

$$T = (M \times M \times \{0\}) \cup (A \times I) \subset M \times M \times I,$$

define $\mu: T \to M$ by

$$\begin{aligned} \mu(x, y, t) &= x && \text{if } y = e, \\ &= y && \text{if } x = e, \\ &= h_t(x) && \text{if } x = y, \\ &= m(x, y) && \text{if } t = 0, \end{aligned}$$

then μ extends, by the homotopy extension theorem, to a map $\mu: M \times M \times I \to M$. Let $m': M \times M \to M$ be defined by $m'(x, y) = \mu(x, y, 1)$ then

$$m'(x, x) = h_1(x) = s'(x) \neq x$$

unless $x = e$, so we have proved the theorem.

The theorem can be proved for H-spaces (X, m, e) where X is a finite polyhedron of a type more general than a manifold essentially just by quoting the results of [4] in their full generality in the proof above. Although Wecken's Theorem is false for finite polyhedra in general, none of the known counterexamples support an H-space structure. Thus it may be that our theorem is true for all finite polyhedra. In any event, the first steps of the proof above do show that, given an H-space (X, m, e) where X is a connected finite polyhedron, there exists $m' \in H_e(X)$ homotopic to m such that (X, m', e) has only a *finite* number of idempotents.

If one employs the techniques of [**6**] and [**1**] in place of those of [**4**], it is possible to prove the theorem for topological manifolds.

REFERENCES

1. R. F. Brown, *On a homotopy converse to the Lefschetz fixed point theorem,* Pacific J. Math. **17** (1966), 407–411. MR **33** #3288.

2. H. Hopf, *Über die algebraische Anzahl von Fixpunkten,* Math. Z. **29** (1929), 493–524.

3. Bo-ju Jiang, *Estimation of the Nielsen numbers,* Acta Math. Sinica **14** (1964), 304–312 = Chinese Math. Acta **5** (1964), 330–339. MR **30** #1510.

4. Gen-hua Shi, *On least number of fixed points and Nielson numbers,* Acta. Math. Sinica **16** (1966), 223–232 = Chinese Math. Acta. **8** (1966), 234–243. MR **35** #1004.

5. F. Wecken, *Fixpunktklassen.* III: *Mindestzahlen von Fixpunkten,* Math. Ann. **118** (1942), 544–577. MR **5,** 275.

6. J. Weier, *Fixpunkttheorie in topologischen Mannigfaltigkeiten,* Math. Z. **59** (1953), 171–190. MR **15,** 337.

UNIVERSITY OF CALIFORNIA, LOS ANGELES