ON ABSOLUTE BOREL-TYPE METHODS OF SUMMABILITY

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1. **Introduction.** Suppose throughout that l, a_n $(n=0, 1, \cdots)$ are arbitrary complex numbers, that $\lambda>0$ and μ is real, and that N is a nonnegative integer such that $\lambda N+\mu\geqq 1$. Let $s_{-1}=0$, $s_n=\sum_{\nu=0}^{n} a_\nu$;

$$a_{\lambda,\mu}(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\lambda n+\mu-1}}{\Gamma(\lambda n+\mu)}, \qquad s_{\lambda,\mu}(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\lambda n+\mu-1}}{\Gamma(\lambda n+\mu)}.$$

Borel-type methods of summability are defined as follows: The series $\sum_0^\infty a_n$ is said to be

(i) summable (B, λ, μ) to l, if $s_{\lambda,\mu}(x)$ is finite for all $x\geqq 0$ and $\lambda e^{-x}s_{\lambda,\mu}(x)\to l$ as $x\to\infty$;

(i)′ summable (B', λ, μ) to l, if $a_{\lambda,\mu}(x)$ is finite for all $x\geqq 0$ and $\int_0^y e^{-x}a_{\lambda,\mu}(x)dx+s_{N-1}\to l$ as $y\to\infty$;

(ii) absolutely summable (B, λ, μ), or summable $|B, \lambda, \mu|$, to l, if the series is summable (B, λ, μ) to l and $e^{-x}s_{\lambda,\mu}(x)$ is of bounded variation on $[0, \infty)$;

(ii)′ absolutely summable (B', λ, μ), or summable $|B', \lambda, \mu|$, to l, if the series is summable (B, λ, μ) to l and $\int_0^y e^{-x}a_{\lambda,\mu}(x)dx$ is of bounded variation on $[0, \infty)$.

Note that the methods $(B, 1, 1)$ and $(B', 1, 1)$ are respectively equivalent to the standard Borel exponential and integral methods B and B'.

The object of this paper is to establish the following absolute summability analogue of a known inclusion theorem for ordinary Borel-type summability ([**2**, Result I] and [**1**, Theorem 2]; see also [**4**]):

THEOREM. *If $\alpha>\lambda$, the series $\sum_0^\infty a_n$ is summable $|B', \alpha, \beta|$ to l, and $a_{\lambda,\mu}(x)$ is finite for all $x\geqq 0$, then the series is summable $|B', \lambda, \mu|$ to l.*

It is known that [**1**, Lemma 4] $a_{\lambda,\mu}(x)$ is finite for all $x\geqq 0$ if and only if $s_{\lambda,\mu}(x)$ is finite for all $x\geqq 0$; and that [**3**, Theorem 17] a series is summable $|B', \lambda, \mu|$ to l if and only if it is summable $|B, \lambda, \mu+1|$ to l. Hence "B'" may be replaced by "B" in the theorem.

2. **Preliminary results.**

LEMMA 1. *If $\delta>0$ and a series is summable $|B', \alpha, \beta|$ to l then it is*

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summable $|B', \alpha, \beta+\delta|$ *to* l.

This lemma is known [5].

LEMMA 2. *If* $\alpha>\lambda$ *and* $\beta/\alpha\geqq\mu/\lambda$, *then there is a function* ψ, *continuous on* $(0, \infty)$, *such that*

$$\frac{\Gamma(\alpha n+\beta)}{\Gamma(\lambda n+\mu)} = \int_0^\infty t^n\psi(t)dt \qquad (n \geqq N), \tag{1}$$

$$\int_0^\infty t^n \,|\, \psi(t) \,|\, dt = O\left(\frac{\Gamma(\alpha n+\beta)}{\Gamma(\lambda n+\mu)}\right) \qquad (n \geqq N), \tag{2}$$

and, for any $\delta>0$,

$$u^{\rho(\alpha-\lambda)}\psi(u^{\alpha-\lambda}) = O(e^{-ku}(u^{1/2}+u^{-\sigma-\delta})) \qquad (0 < u < \infty) \tag{3}$$

where $\rho=1-(\beta-\mu)/(\alpha-\lambda)$, $\sigma=\beta-\alpha\mu/\lambda$, $k=((\alpha-\lambda)/\lambda)(\lambda/\alpha)^{\alpha/(\alpha-\lambda)}$.

PROOF. Let $h(s)=\Gamma(\alpha s+\beta)/\Gamma(\lambda s+\mu)$. Then by Stirling's theorem (see [2, p. 129]), there is a positive constant C such that

$$h(s) = e^{(\alpha\log\alpha-\lambda\log\lambda-\alpha+\lambda)s}s^{(\alpha-\lambda)s+\beta-\mu}\{C+O(1/|s|)\}$$

when $|s|$ is large and $\operatorname{Re} s>-\mu/\lambda$. Since $N>-\mu/\lambda$, it follows from the proof of Lemma 4 in [2], with $\sigma_0=-\mu/\lambda$, $\nu=N$, that there is a function ϕ, continuous on $(0, \infty)$, such that

$$h(n) = \int_0^\infty t^{n-N}\phi(t)dt \qquad (n \geqq N);$$

$$\int_0^\infty t^{n-N} \,|\, \phi(t) \,|\, dt = O(h(n)) \qquad (n \geqq N);$$

$$t^{-N}\phi(t) = O(t^{\mu/\lambda-1-\delta/(\alpha-\lambda)}) = O(t^{-(\sigma+\delta)/(\alpha-\lambda)}) \qquad \text{as} \qquad t\to 0+;$$

and

$$t^{-N}\phi(t) \sim Ke^{-kt^{1/(\alpha-\lambda)}}t^{-\rho+1/2(\alpha-\lambda)} \qquad \text{as} \qquad t\to\infty,$$

where K is a positive constant.

Putting $\psi(t)=t^{-N}\phi(t)$, we obtain the conclusions of Lemma 2.

3. **Proof of the theorem.** Let

$$\gamma=\alpha/\lambda, \qquad \rho=1-(\beta-\mu)/(\alpha-\lambda), \qquad \sigma=\beta-\gamma\mu,$$
$$k=(\gamma-1)\gamma^{\gamma/(1-\gamma)}, \qquad \delta=(\gamma-1)^2/\gamma.$$

By Lemma 1, we may suppose, without loss in generality that $\beta\geqq\gamma\mu$, i.e. that $\sigma\geqq 0$.

The main hypotheses of the theorem are that

$$\int_0^\infty e^{-y} \mid a_{\alpha,\beta}(y) \mid dy < \infty, \tag{4}$$

and that

$$a_{\lambda,\mu}(x) \text{ is finite for all } x \geqq 0. \tag{5}$$

Let ψ be the function specified in Lemma 2. Then, for $0<x<\infty$,

$$\begin{aligned} a_{\lambda,\mu}(x) &= \sum_{n=N}^{\infty} \frac{a_n x^{\lambda n+\mu-1}}{\Gamma(\alpha n+\beta)} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\lambda n+\mu)} = \sum_{n=N}^{\infty} \frac{a_n x^{\lambda n+\mu-1}}{\Gamma(\alpha n+\beta)} \int_0^\infty t^n \psi(t) dt \\ &= x^{\mu-1+(1-\beta)/\gamma} \int_0^\infty t^{(1-\beta)/\alpha} \psi(t) dt \sum_{n=N}^{\infty} \frac{a_n (x^{1/\gamma} t^{1/\alpha})^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \\ &= x^{\mu-1+(1-\beta)/\gamma} \int_0^\infty t^{(1-\beta)/\alpha} \psi(t) a_{\alpha,\beta}(x^{1/\gamma} t^{1/\alpha}) dt, \end{aligned} \tag{6}$$

the inversion of sum and integral being legitimate since, by (2), there is a constant M such that

$$\sum_{n=N}^{\infty} \frac{\mid a_n \mid x^{\lambda n+\mu-1}}{\Gamma(\alpha n+\beta)} \int_0^\infty t^n \mid \psi(t) \mid dt < M \sum_{n=N}^{\infty} \frac{\mid a_n \mid x^{\lambda n+\mu-1}}{\Gamma(\lambda n+\mu)},$$

which is finite by (5).

Substitute $t=x^{-\lambda}y^{\alpha}$, $dt=\alpha x^{-\lambda}y^{\alpha-1}dy$ in the final integral in (6) to get

$$a_{\lambda,\mu}(x) = \alpha x^{\mu-\lambda-1} \int_0^\infty y^{\alpha-\beta} a_{\alpha,\beta}(y) \psi(x^{-\lambda}y^{\alpha}) dy \qquad (0 < x < \infty),$$

and hence

$$\begin{aligned} &\int_0^\infty e^{-x} \mid a_{\lambda,\mu}(x) \mid dx \\ &\qquad \leqq \alpha \int_0^\infty \mid a_{\alpha,\beta}(y) \mid y^{\alpha-\beta} dy \int_0^\infty e^{-x} x^{\mu-\lambda-1} \mid \psi(x^{-\lambda}y^{\alpha}) \mid dx. \end{aligned} \tag{7}$$

Now substitute $x=yv^{\gamma-1}$, $dx=(\gamma-1)yv^{\gamma-2}dv$ in the inner integral on the right-hand side of (7) to get

$$\begin{aligned} &\int_0^\infty e^{-x} \mid a_{\lambda,\mu}(x) \mid dx \\ &\leqq \alpha(\gamma-1) \int_0^\infty \mid a_{\alpha,\beta}(y) \mid dy \int_0^\infty e^{-yv^{\gamma-1}} v^{-\sigma-1} (y/v)^{\rho(\alpha-\lambda)} \mid \psi((y/v)^{\alpha-\lambda}) \mid dv. \end{aligned}$$

Consequently, by (3), there is a constant M_1 such that

$$\int_0^\infty e^{-x} \mid a_{\lambda,\mu}(x) \mid dx \leqq M_1 \int_0^\infty e^{-y} \mid a_{\alpha,\beta}(y) \mid I(y) dy \tag{8}$$

where

$$I(y) = \int_0^\infty e^{-(v^\gamma - v + k)y/v} \left\{ (y/v)^{1/2} + (y/v)^{-\sigma-\delta} \right\} v^{-\sigma-1} dv.$$

Let $f(v) = v^\gamma - v + k$, $c = \gamma^{1/(1-\gamma)}$. Then $f(c) = f'(c) = 0$, $f(v) > 0$ when $v > 0$, $v \neq c$, and

$$f(v)/(v-c)^2 \to f''(c)/2 = \gamma(\gamma - 1)c^{\gamma-2}/2 \quad \text{as} \quad v \to c.$$

Hence there are positive constants p, q, r such that

$$f(v) \geqq p, \qquad v^{-\gamma} f(v) \geqq q \quad \text{when } 0 < v < c/2 \quad \text{or} \quad v > 3c/2;$$

and $f(v) \geqq rv(v-c)^2$ when $c/2 < v < 3c/2$.

It follows that, for $y > 0$,

$$\begin{aligned} I(y) &\leqq y^{1/2} \int_{c/2}^{3c/2} e^{-r(v-c)^2} v^{-\sigma-3/2} dv + y^{1/2} \int_0^\infty e^{-py/v} v^{-\sigma-3/2} dv \\ &\quad + y^{-\sigma-\delta} \int_{c/2}^{3c/2} v^{\delta-1} dv + y^{-\sigma-\delta} \int_0^\infty e^{-qyv^{\gamma-1}} v^{\delta-1} dv \\ &\leqq 2\left(\frac{c}{2}\right)^{-\sigma-3/2} y^{1/2} \int_0^{c/2} e^{-rt^2} dt + y^{-\sigma} \int_0^\infty e^{-pt} t^{\sigma-1/2} dt \\ &\quad + y^{-\sigma-\delta} \left(\frac{3c}{2}\right)^\delta \delta^{-1} + y^{-\sigma-\delta\gamma/(\gamma-1)} \int_0^\infty e^{-qt} t^{-1+\delta/(\gamma-1)} dt \\ &\leqq M_2(1 + y^{-\sigma} + y^{-\sigma-\delta} + y^{-\sigma-\gamma+1}) \end{aligned}$$

where M_2 is a constant; i.e.

$$I(y) = O(1) \qquad (1 \leqq y < \infty), \tag{9}$$

and, since $\delta = (\gamma-1)^2/\gamma < \gamma - 1$,

$$I(y) = O(y^{-\sigma-\gamma+1}) \qquad (0 < y < 1). \tag{10}$$

In virtue of (10), we have

$$\begin{aligned} I(y) a_{\alpha,\beta}(y) &= I(y) \sum_{n=N}^\infty \frac{a_n y^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)} \\ &= O(y^{-\sigma-\gamma+1+\alpha N+\beta-1}) = O(y^{\gamma(\lambda N+\mu-1)}), \\ &= O(1) \quad (0 < y < 1). \end{aligned} \tag{11}$$

It follows from (4), (9) and (11) that

$$\int_0^\infty e^{-y} \left| a_{\alpha,\beta}(y) \right| I(y)dy < \infty.$$

Consequently, by (8),

$$\int_0^\infty e^{-x} \left| a_{\lambda,\mu}(x) \right| dx < \infty,$$

i.e. $\sum_0^\infty a_n$ is summable $|B, \lambda, \mu|$.

Further, by the inclusion theorem for ordinary Borel-type summability referred to in §1, the $|B, \lambda, \mu|$ sum of the series $\sum_0^\infty a_n$ is the same as its $|B, \alpha, \beta|$ sum. This completes the proof.

References

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