JORDAN STRUCTURES IN SIMPLE GRADED RINGS

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1. **Introduction.** In a recent paper [3] we proved graded analogs to theorems of Herstein [1], [2] about the Lie structure of a simple ring. In this note results about the Jordan structure of a simple graded ring will be given. The main results are Theorem 1, which states that a homogeneous Jordan ideal that contains an even element also contains an irrelevant ideal, and Theorem 2, which states that a homogeneous Jordan ideal that is also a subring must contain an irrelevant ideal.

2. **Preliminaries.** In a graded ring $R=\oplus_{i\geqq 0}R_i$, ideals of the form $\oplus_{i\geqq n} R_i$ are *irrelevant*. R is a simple graded ring (sgr) if $R_iR_j\neq(0)$ for all i and j, and R has no relevant homogeneous ideals. If $x\in R_\alpha$ and $y\in R_\beta$, then $[x, y]=xy-(-1)^{\alpha\beta}yx((x, y)=xy+(-1)^{\alpha\beta}yx)$ is called their *Lie product* (*Jordan product*). The *center* of R is $Z(R) =\{x: [x, y]=0 \text{ for all } y\in R\}$.

PROPOSITION 1. *Let $R= \oplus_{i\geqq 0} R_i$ be a sgr. If $0\neq a\in R_j$, then $R_iaR_k =R_{i+j+k}$ for all i and k. If b is homogeneous and $R_0bR_0=(0)$, then $b=0$.*

A proof may be found in [3].

3. **Lemma.** *Let R be a graded ring and let U be a homogeneous Jordan ideal of R. If $a, b\in U$ are homogeneous, then for all homogeneous $x\in R$ we have $[(a, b), x]\in U$.*

PROOF. $(a, [x, b])-([a, x], b)=(-1)^{ax}[x, (a, b)]$. The left side of the equation is an element of U, so the result follows.

THEOREM 1. *Let $R= \oplus_{i\geqq 0} R_i$ be a sgr of characteristic $\neq 2$ and let U be a homogeneous Jordan ideal of R. If U contains a nonzero even element of R, then U contains a nonzero irrelevant ideal of R.*

PROOF. Let $a, b\in U$ $x\in R$ be homogeneous. Then $[(a, b), x]\in U$ and $((a,b),x)\in U$ imply $2x(a,b)\in U$ which in turn implies that $(2x(a,b),y)\in U$ for all homogeneous y. Thus, $2R_i(a, b)R_j\subseteq U$ for all i and j.

If $2R_i(a, b)R_j=(0)$ for all i and j, then by Proposition 1 $(a, b)=0$, and so in this case $(U, U)=(0)$. If $0\neq a\in U$ is even, $0=(a, (a, x))$ implies $2axa=0$ for all homogeneous x. Thus, $a=0$, a contradiction.

Hence, there exist i and j such that $0\neq 2R_i(a, b)R_j\subseteq U$. Therefore, $U\supseteq \oplus_{k\geqq a+b+i+j} R_k\neq(0)$.

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PROPOSITION 2. *Let R be a* sgr, char $R\neq 2$, *and let U be a homogeneous Jordan ideal of R that does not contain a nonzero irrelevant ideal of R. If $a\in U$ satisfies $[a, R]\subseteq U$, then $a=0$.*

PROOF. Let x and y be homogeneous. If $a\neq 0$, then $[a, x]\in U$ and $(a, x)\in U$ imply $ax\in U$, so $(ax, y)\in U$. Thus, $yax\in U$, so $U\supseteq \bigoplus_{i\geq a} R_i$.

COROLLARY. *With R and U as above, $U\cap Z(R)=(0)$.*

THEOREM 2. *Let R be a* sgr, char $R\neq 2$, *and let U be a homogeneous Jordan ideal and a subring of R. Either $U=(0)$ or U contains a nonzero irrelevant ideal of R.*

PROOF. If $(U, U)=(0)$, $a=0$ for all even $a\in U$. If $0\neq a\in U$ is odd, $a^2=0$ and so $a(a, x)=0$ for all even $x\in R$. Hence, $axa=0$, so $a=0$. Thus, $(U, U)=(0)$ implies that $U=(0)$.

If $(U, U)\neq(0)$ let a and $b\in U$, $(a, b)\neq 0$, and let c be homogeneous. Then $(ab, c)=(a, b)c-(-1)^{ab}b(a, c)+(-1)^{a(b+c)}(b, ca)$. Thus, $(a, b)R_i\subseteq U$ for all i. Let d be homogeneous. Then $(d, (a, b)c)$ $=d(a, b)c+(-1)^{d(a+b+c)}(a, b)cd\in U$. Now,

$$(a, bcd) = abcd + (-1)^{a(b+c+d)}bcda \in U.$$

An examination of the parities involved shows that $(a, bcd)=(a, b)cd$ $\pm(ba, cd)\pm(cd, b)a$. Thus $(a, b)cd\in U$, and so $d(a, b)c\in U$. This implies that $U\supseteq \bigoplus_{i\geq a+b} R_i$.

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