LINEARLY ORDERED COLLECTIONS AND PARACOMPACTNESS

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1. Introduction. The purpose of this paper is to characterize paracompactness by conditions which are formally weaker than some considered by Michael [3], [4], [5], and which are related to some considered by Tamano [7], [8], and Katuta [2]. In addition some consequences of these concepts are derived. The main result is the following

THEOREM 1. *Let X be a regular space. The following are equivalent*

(a) *X is paracompact.*

(b) *Every open cover of X has an open refinement which is linearly locally finite.*

(c) *Every open cover of X has an open refinement which is linearly closure-preserving.*

(d) *Every open cover of X has an open refinement which is linearly cushioned in it.*

Let us now define all of the terms which are mentioned in Theorem 1. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of a topological space. A collection $\mathcal{U}$ endowed with a linear (=total) order is said to be *linearly locally finite with respect to* $\leqq$ provided that every majorized subcollection (that is, every subcollection of $\mathcal{U}$ having an upper bound with respect to $\leqq$) is locally finite. This definition is equivalent to that used by H. Tamano in [7] where he proved that (a) and (b) in Theorem 1 are equivalent in completely regular spaces. A collection $\mathcal{U}$ endowed with a linear order is said to be *linearly closure-preserving with respect to* $\leqq$ provided that every majorized subcollection of $\mathcal{U}$ is closure-preserving. In order to define the term mentioned in (d), we first restate a definition given in [5]. A collection $\mathcal{U}$ is said to be *cushioned in a collection* $\mathcal{V}$ *with cushion map* $f\colon \mathcal{U}\to\mathcal{V}$ provided for every subcollection $\mathcal{U}'$ of $\mathcal{U}$ we have $\mathrm{cl}(\bigcup\mathcal{U}')\subset\bigcup f(\mathcal{U}')$. We say that a collection $\mathcal{U}$ endowed with a linear order is *linearly cushioned in a collection* $\mathcal{V}$ *with cushion map* $f\colon\mathcal{U}\to\mathcal{V}$ provided for every majorized subcollection $\mathcal{U}'$ of $\mathcal{U}$ we have $\mathrm{cl}(\bigcup\mathcal{U}')\subset\bigcup f(\mathcal{U}')$. We will omit explicit mention of the cushion map if no confusion will result.

The above definitions of linearly closure-preserving and "linearly cushioned" differ from those given by H. Tamano in [8] where he

Presented to the Society August 30, 1968; received by the editors September 9 1968.

required the linear order to be a well-order. In that paper he proved (using his definitions) that (a), (c), and (d) of Theorem 1 are equivalent in a completely regular space. The proof of Theorem 1 is given in §2, and is different from Tamano's proof because he made use of the Stone-Čech compactification of completely regular spaces. This method, of course, is not available for arbitrary regular spaces.

In §3 we consider the relationship of the "linear concepts" to the concepts considered by Michael. In §4 we consider the consequences of replacing the word "majorized" by the word "bounded" in the above definitions. In §5 we discuss the concept of order local finiteness in the sense of Katuta, and give an analogue of Theorem 1.

2. **Proof of Theorem 1.** We begin with a key set-theoretic lemma.

LEMMA 1. *Every set with a linear order* $\leqq$ *can be given a well-ordering* $\leqq\leqq$ *such that every* $\leqq\leqq$ *majorized set is also* $\leqq$ *majorized.*

PROOF. Let $\leqq$ be a linear order on a set X. Let B be a well-ordered cofinal subset of X. If B has a largest element, then any well-order on X will work. We assume B has no largest element with respect to $\leqq$. For every $b \in B$ define $A_b = \{x \in X : x < b\}$, and

$$D_b = A_b - \bigcup \{A_p : p < b \quad \text{and} \quad p \in B\}.$$

Notice that $\{D_b : b \in B\}$ is a partition of X. Now let $\leqq_b$ be any well-order on D_b for every $b \in B$, and define an order $\leqq\leqq$ on X as follows. Let $x, y \in X$, then $x \leqq\leqq y$ if and only if

(1) there exists $b \in B$ such that $x, y \in D_b$ and $x \leqq_b y$, or

(2) there exist $b, c \in B$ such that $x \in D_b$, $y \in D_c$ and $b < c$.

It is routine to check that $\leqq\leqq$ is a well-order for X having the desired property.

The author does not know if every linearly closure-preserving open cover of a topological space has a closure-preserving refinement. For the other kinds of collections, however, one can prove the following

LEMMA 2. *Let X be a topological space. Every open cover $\mathfrak{U}$ of X which is linearly locally finite (resp. linearly cushioned in $\mathfrak{V}$) has a refinement —not necessarily open—which is locally finite (resp. cushioned in $\mathfrak{V}$).*

PROOF. We prove the second case. Let $\mathfrak{U}$ be linearly cushioned in $\mathfrak{V}$ with respect to $\leqq$. Let $\leqq\leqq$ be a well-order on $\mathfrak{U}$ such that every $\leqq\leqq$ majorized subset of $\mathfrak{U}$ is $\leqq$ majorized. Clearly, $\mathfrak{U}$ is linearly cushioned in $\mathfrak{V}$ with respect to $\leqq\leqq$ and with the same cushion map $f: \mathfrak{U} \to \mathfrak{V}$. Define $H_U = U - \bigcup \{V \in \mathfrak{U} : V << U\}$ for every $U \in \mathfrak{U}$. Let $\mathfrak{H} = \{H_U : U \in \mathfrak{U}\}$, then $\mathfrak{H}$ is a refinement of $\mathfrak{U}$ and is cushioned in $\mathfrak{V}$

with cushion map $g: \mathcal{H} \to \mathcal{V}$ defined by $g(H_U) = f(U)$. To see that $\mathcal{H}$ covers X, let $x \in X$, and let U be the first element of $\mathcal{U}$ containing x, then $x \in H_U$. To see that $\mathcal{H}$ is cushioned in $\mathcal{V}$, let $\mathcal{H}'$ be any subcollection of $\mathcal{H}$, and let p be in $\mathrm{cl}(\bigcup \mathcal{H}')$. Since $\mathcal{U}$ is an open cover of X, there exists $U \in \mathcal{U}$ such that $p \in U$. Then U is an open set which misses H_V for all $V >> U$. Set $\mathcal{H}'' = \{H_V \colon V \leqq\leqq U\}$, and note that $p \in \mathrm{cl}(\bigcup \mathcal{H}'')$. The subcollection $\mathcal{U}'' = \{V \in \mathcal{U} \colon H_V \in \mathcal{H}''\}$ is majorized by U hence $\mathrm{cl}(\bigcup \mathcal{U}'') \subset \bigcup f(\mathcal{U}'') = \bigcup g(\mathcal{H}'')$. It follows that $p \in \bigcup g(\mathcal{H}')$, and this implies that $\mathrm{cl}(\bigcup \mathcal{H}') \subset \bigcup g(\mathcal{H}')$.

Proof of Theorem 1. The implications (a)⇒(b)⇒(c)⇒(d) are obvious. It remains to prove (d)⇒(a). By Theorem 1.1 of [5], it suffices to show that every open cover has a refinement which is cushioned in it, but this follows immediately from Lemma 2.

3. **Relations of linearly ordered collections to some other properties.** It is clear that every σ-locally finite collection is a linearly locally finite collection. Conversely, we have the following obvious

Proposition 1. *Let $\mathcal{U}$ be a collection which is linearly locally finite in a space X.*

(a) *If $\mathcal{U}$ has a largest element, then $\mathcal{U}$ is locally finite.*

(b) *If $\mathcal{U}$ has a countable cofinal subset, then $\mathcal{U}$ is σ-locally finite.*

(c) *If $\mathcal{U}$ does not have a countable cofinal subset, then $\mathcal{U}$ is point-finite (i.e., every point in X is a member of only finitely many elements of $\mathcal{U}$).*

Remark. Proposition 1 (a) and (b) remain true if everywhere the words "locally finite" are replaced by "closure-preserving," or "cushioned in $\mathcal{V}$."

Proposition 2. *In a space X which satisfies the first axiom of countability, every collection which is linearly locally finite is σ-locally finite.*

Proof. Let $\mathcal{U}$ be linearly locally finite with respect to a linear order $\leqq$. Assume $\mathcal{U}$ is not σ-locally finite; so $\mathcal{U}$ does not have a countable cofinal subset. Since $\mathcal{U}$ is not locally finite there exists $x \in X$ such that every neighborhood of x intersects infinitely many members of $\mathcal{U}$. Let $\{N_i(x) \colon i = 1, 2, \cdots\}$ be a fundamental system of neighborhoods of x such that $N_i(x) \supset N_{i+1}(x)$. Then there exists an infinite sequence $\mathcal{U}' = \{U_i \colon i = 1, 2, \cdots\}$ of distinct elements of $\mathcal{U}$ such that $U_i \cap N_i(x) \neq \varnothing$. Since $\mathcal{U}'$ is not cofinal in $\mathcal{U}$ it is majorized, and therefore locally finite. Thus, there exists $N_i(x)$ which intersects only finitely many elements of $\mathcal{U}'$. But this is impossible since $N_i(x) \cap U_n \neq \varnothing$ for $n \geqq i$.

Remark. Proposition 2 remains true if everywhere the words "locally finite" are replaced by "closure-preserving" or "cushioned in

$\mathcal{V}$." The referee pointed out that a similar proof can be given to show that Proposition 2 holds in a T_1-space which is a q-space in the sense of E. Michael [6]. Further, one can show that in a T_1-regular q-space every linearly closure-preserving collection is σ-closure-preserving. It is not possible, however, to extend Proposition 2 to q-spaces and cushioned collections. Here is a simple example. Let Ω be the first uncountable ordinal, then $[0, \Omega]$ with the order topology is a compact Hausdorff space (hence a q-space). Let $\mathcal{V} = \{[0, \Omega)\}$, and let $\mathcal{U} = \{\{p\} : p \in [0, \Omega)\}$. It is easy to see that $\mathcal{U}$ is linearly cushioned in $\mathcal{V}$ with respect to the usual order on $[0, \Omega)$, but $\mathcal{U}$ is not σ-cushioned in $\mathcal{V}$.

An immediate consequence of the preceding propositions is the following modification of the Nagata-Smirnov Theorem. Call a base for a topology a *σ-linearly locally finite base* if it is a countable union of linearly locally finite collections.

COROLLARY. *A regular T_1-space is metrizable if and only if its topology has a σ-linearly locally finite base.*

We now partially strengthen Lemma 2 for normal spaces.

PROPOSITION 3. *Let X be a normal space. Every linearly locally finite open cover of X has a σ-locally finite open refinement.*

PROOF. Let $\mathcal{U}$ be a linearly locally finite open cover of X. If $\mathcal{U}$ is σ-locally finite, then there is nothing to prove. If $\mathcal{U}$ is not σ-locally finite, then by Proposition 1 it is point-finite. This means that $\mathcal{U}$ is shrinkable, i.e., there exists an open cover $\{H_U: U \in \mathcal{U}\}$ of X such that $\overline{H}_U \subset U$ for all $U \in \mathcal{U}$. Let $W_U = U - \bigcup\{\overline{H}_V: V < U\}$. Then $\mathcal{W} = \{W_U: U \in \mathcal{U}\}$ is in fact a locally finite open refinement of $\mathcal{U}$.

REMARK. In light of Proposition 3 it is, perhaps, interesting to note that every σ-locally finite open cover of a normal T_2-space has a locally finite open refinement if and only if every normal T_2-space is countably paracompact. Whether every normal T_2-space is countably paracompact is a well-known unsolved problem raised by C. H. Dowker.

From Theorem 1 we see that if every open cover of a regular space has a linearly locally finite open refinement, then the space is paracompact. The following question arises: If every open cover of a regular space has a σ-linearly locally finite open refinement (i.e., a refinement which is a countable union of linearly locally finite open collections), then is the space paracompact? The following example shows that the answer is in the negative.

EXAMPLE 1. A completely regular T_2-space X which is not normal (hence not paracompact), and every open cover of X has a σ-linearly locally finite open refinement. This example is due to J. Dieudonné [1]. Let ω, Ω be the first infinite and the first uncountable ordinals respectively. Let $X=[0, \Omega]\times[0, \omega]-(\Omega, \omega)$. For every $n<\omega$ let $\{U_{x,n}=[x, \Omega]\times\{n\}: x<\Omega\}$ be a fundamental system of neighborhoods of (Ω, n). Let $\{V_{x,n}=\{x\}\times[n, \omega]: n<\omega\}$ be a fundamental system of neighborhoods of (x, ω). All other points (x, n) for $x<\Omega$, $n<\omega$ are to be isolated. Let $\mathfrak{U}$ be any open cover of X. For every $n<\omega$ (resp. $x<\Omega$) let $U_n=U_{x,n}$ (resp. $V_x=V_{x,n}$) be a basic open neighborhood of (Ω, n) (resp. (x, ω)) which is contained in some element of $\mathfrak{U}$. Let $\mathfrak{V}_1=\{U_n: n\in[0, \omega)\}$, and order $\mathfrak{V}_1$ by the usual order on $[0, \omega)$. Let $\mathfrak{V}_2=\{V_x: x\in[0, \Omega)\}$, and order $\mathfrak{V}_2$ by the usual order on $[0, \Omega)$. Let $\mathfrak{V}_3$ be the set of all points which are not contained in any element of $\mathfrak{V}_1\cup\mathfrak{V}_2$, and give $\mathfrak{V}_3$ any well-order. Clearly, $\bigcup_{i=1}^{3}\mathfrak{V}_i$ is an open σ-linearly locally finite refinement of $\mathfrak{U}$.

4. **Majorized versus bounded.** The main purpose of this section is to show that the conditions in Theorem 1 cannot be weakened by replacing the word "majorized" by "bounded (above and below)" in the definitions given in §1. To facilitate further discussion we make the following definition: A collection $\mathfrak{U}$ endowed with a linear order $\leqq$ is said to be *weakly linearly locally finite with respect to* $\leqq$ provided every bounded subcollection of $\mathfrak{U}$ is locally finite. It is clear that every linearly locally finite collection is weakly linearly locally finite. If there is a least element for the linear order, then the converse holds. In any case we have

PROPOSITION 4. *A collection $\mathfrak{U}$ is weakly linearly locally finite if and only if $\mathfrak{U}=\mathfrak{U}_1\cup\mathfrak{U}_2$ with each of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ linearly locally finite.*

PROOF. To show the only if part, let $U_0\in\mathfrak{U}$ and define

$$\mathfrak{U}_1 = \{U\in\mathfrak{U}: U_0 \leqq U\}.$$

Give $\mathfrak{U}_1$ the induced order from $\mathfrak{U}$. Let $\mathfrak{U}_2=\mathfrak{U}-\mathfrak{U}_1$, and give $\mathfrak{U}_2$ the order induced from the inverse of the original order relation on $\mathfrak{U}$. Then $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are linearly locally finite.

To prove the if part, let $\mathfrak{U}_i$ be linearly locally finite with respect to $\leqq_i$ for $i=1, 2$, and let $\mathfrak{U}_2''=\mathfrak{U}_2-\mathfrak{U}_1$. Define a linear order on $\mathfrak{U}$ as follows. If U, $U'\in\mathfrak{U}$, then $U\leqq U'$ if and only if (1) $U\in\mathfrak{U}_2''$ and $U'\in\mathfrak{U}_1$, (2) U, $U'\in\mathfrak{U}_1$ and $U\leqq_1 U'$, or (3) U, $U'\in\mathfrak{U}_2''$ and $U'\leqq_2 U$. Then $\mathfrak{U}$ is weakly linearly locally finite with respect to $\leqq$.

The space X of Example 1 is a nonparacompact space in which every open cover has a weakly linearly locally finite open refinement. This is easy to see because $\mathfrak{V}_3$ in that example is actually locally finite. If we let $\mathfrak{V}_1' = \mathfrak{V}_3 \cup \mathfrak{V}_1$, and order $\mathfrak{V}_1'$ in such a way that every element of $\mathfrak{V}_3$ precedes every element of $\mathfrak{V}_1$, then $\mathfrak{V}_1'$ is also linearly locally finite. By Proposition 4, $\mathfrak{V} = \mathfrak{V}_1' \cup \mathfrak{V}_2$ is weakly linearly locally finite.

REMARK. In a similar manner one may define "weakly linearly closure-preserving" and "weakly linearly cushioned in $\mathfrak{V}$". The obvious analogues of Proposition 4 still hold.

5. **Order local finiteness.** Y. Katuta [2] called a collection $\mathfrak{U}$ endowed with a linear order $\leqq$ *order locally finite with respect to* $\leqq$ provided for every $U \in \mathfrak{U}$ the collection $\{V \in \mathfrak{U}: V \leqq U\}$ is locally finite at each point of U. He also proved [2, Lemma 2, p. 616] that a regular space is paracompact if and only if every open cover has an open order locally finite refinement. This is a stronger result than (b)$\Rightarrow$(a) in Theorem 1, and his proof did not use well-ordering. It is natural to ask if all of Theorem 1 can be similarly strengthened. The answer is in the negative for the following extension of Katuta's concept. We say that a collection $\mathfrak{U}$ endowed with a linear order $\leqq$ is *order closure-preserving* provided for every $U \in \mathfrak{U}$ and every $x \in U$ and every subcollection $\mathfrak{U}' \subset \{V \in \mathfrak{U}: V \leqq U\}$, if $x \in \mathrm{cl}(\bigcup \mathfrak{U}')$, then there exists a $U' \in \mathfrak{U}'$ such that $x \in \mathrm{cl}(U')$.

EXAMPLE 2. Let $X = [0, \Omega)$ with the order topology. Every open cover of X has an open order closure-preserving refinement, but X is not paracompact. Let $\mathfrak{V}$ be an open cover of X, and let $x \in X$. If x is a limit ordinal let W_x be any open interval containing x which is contained in some element of $\mathfrak{V}$. If x is an isolated point let $W_x = \{x\}$. Then $\mathfrak{W} = \{W_x: x \in [0, \Omega)\}$ is order closure-preserving with respect to the inverse of the usual order on the index set $[0, \Omega)$, and clearly an open refinement of $\mathfrak{V}$.

It is possible, however, to get some results if we consider well-ordered collections. Call a collection *well-ordered closure-preserving* if it is order closure-preserving with respect to a well-order. A collection $\mathfrak{U}$ endowed with a well-order $\leqq$ is said to be *well-ordered cushioned in a collection* $\mathfrak{V}$ *with cushion map* $f: \mathfrak{U} \to \mathfrak{V}$ provided for every $U \in \mathfrak{U}$ and for every $x \in U$ and every subcollection $\mathfrak{U}' \subset \{V \in \mathfrak{U}: V \leqq U\}$ if $x \in \mathrm{cl}(\bigcup \mathfrak{U}')$, then $x \in \bigcup f(\mathfrak{U}')$. We can now incorporate Katuta's result in the following theorem.

THEOREM 2. *Let X be a regular space. The following are equivalent.*

(a) *X is paracompact.*

(b) *Every open cover of X has an open refinement which is order locally finite.*

(c) *Every open cover of X has an open refinement which is well-ordered closure-preserving.*

(d) *Every open cover of X has an open refinement which is well-ordered cushioned in it.*

PROOF. Katuta proved that (a) and (b) are equivalent. Clearly (a)$\Rightarrow$(c)$\Rightarrow$(d). The proof that (d)$\Rightarrow$(a) is similar to the proof that (d)$\Rightarrow$(a) in Theorem 1.

The author would like to thank the referee for bringing Katuta's results to his attention, and for other helpful comments.

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