A NOTE ON A THEOREM BY H. D. BRUNK[1]

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Throughout the paper, the notation will be consistent with that used by H. D. Brunk in [1]. That is $(\Omega, \mathcal{S}, \mu)$ is a complete measure space, and L_2 denotes the set of square integrable functions corresponding to it. We shall call $\mathcal{L}$, a collection of sets in $\mathcal{S}$, a sub-σ-lattice if it is closed under countable unions and intersections, and contains the null set $\varnothing$, and Ω. A function X is $\mathcal{L}$-measurable if $[X>a]\in\mathcal{L}$ for all real a. $L_2(\mathcal{L})$ denotes the set of $\mathcal{L}$-measurable functions which are also in L_2. A family C of measurable functions is called a convex cone if $k\geqq 0$, $X\in C$, $Y\in C\Rightarrow kX\in C$, $X+Y\in C$. A collection of functions is a lattice if the pointwise supremum and infinum of any two functions in the collection is in the collection. If M is a collection of functions, $-M=\{-X: X\in M\}$. Similarly, $\mathcal{L}^c=\{A: A^c\in\mathcal{L}\}$. I_A or $I(A)$ will be the indicator function of the set A.

In [1], H. D. Brunk stated the following theorem.

THEOREM. *M, a subset of L_2 is $L_2(\mathcal{L})$ for some σ-lattice $\mathcal{L}$ containing Φ and Ω if and only if*

(1) *M is a lattice closed in L_2;*

(2) *a real, $X\in M$, $A=[X>a]$, $\mu(A)<\infty$ implies $I(A)\in M$; a real, $X\in M$, $A=[X\geqq a]$, $\mu(A^c)<\infty$ implies $-I(A^c)\in M$;*

(3) *M is a convex cone.*

However the theorem is not quite true as stated, for if M is the set of nonnegative functions which are also square-integrable with respect to the measure space of the reals, Borel sets, and Lebesque measure, then M satisfies the conditions of the theorem, yet M is not $L_2(\mathcal{L})$ for any σ-lattice $\mathcal{L}$.

However, if we slightly change (2), we can drop the requirement that M be a lattice to obtain the following theorem.

THEOREM. *M, a nonempty subset of L_2, is $L_2(\mathcal{L})$ for some σ-lattice $\mathcal{L}$ containing Φ and Ω if and only if*

(1) *M is a convex cone closed in L_2;*

(2) *$a\geqq 0$, $X\in M(-M)$, $A=[X\geqq a]$, $\mu(A)<\infty$ implies $I(A)\in M(-M)$.*

Received by the editors August 16, 1968.

[1] This investigation was supported in part by PHS Research Grant No. GM 14448-01 from the Division of General Medical Sciences.

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PROOF. (The proof incorporates many of the arguments given in Brunk's proof as well as the notation used in [1].) To show necessity, note that it is easy to verify that $L_2(\mathfrak{L})$ is a convex cone since $[X+Y>a]=\bigcup_r \{[X>a-r]\cap[Y>r]\}$ where the union is taken over the set of rational numbers. To show that $L_2(\mathfrak{L})$ is closed in L_2, assume that $f_n \xrightarrow{L_2} f$, where $f_n \in L_2(\mathfrak{L})$ for all n. Then there exists, a subsequence f_{n_j} such that $f_{n_j} \xrightarrow{\text{a.e.}} f$. Now for each real number a,

$$[f > a] = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} [f_{n_j} > a + 1/m],$$

which belongs to $\mathfrak{L}$ since $\mathfrak{L}$ is closed under countable unions and intersections. Thus $f \in L_2(\mathfrak{L})$, so that $L_2(\mathfrak{L})$ is closed in L_2. It is easy to verify that $L_2(\mathfrak{L})$ also satisfies (2).

Let us now be concerned with showing the sufficiency. Observe first of all that $\mu([X>a])<\infty$, where $X\in M$ and $a\geqq 0$, implies that $I([X>a])\in M$ since $I([X\geqq a+1/n])\in M$ for all n by (2) and converges to $I([X>a])$ in L_2.

Now let $\mathfrak{L}^+=\{[X>a];\, a\geqq 0,\, X\in M\}$ and

$$\mathfrak{L}^- = \{[X \geqq -a]; \qquad a \geqq 0,\ X \in M].$$

We will show that $\mathfrak{L}^+$ is closed under countable unions and countable intersections. Let C be a countable union of sets in $\mathfrak{L}^+$. Then since $[X>0]=\bigcup_{n=1}^{\infty}[X>1/n]$, we may assume that $C=\bigcup_{i=1}^{\infty} C_i$ where $0<\mu(C_i)<\infty$ and $C_i\in\mathfrak{L}^+$. Define $Y_n=\sum_{i=1}^{n} I(C_i)[2^i\mu(C_i)\vee 2^i]^{-1}$ where $2^i\mu(C_i)\vee 2_i$ denotes the supremum of $2^i\mu(C_i)$ and 2^i. Then $Y_n\in M$ for all n by (1). Since $\{Y_n\}$ is a monotone nondecreasing sequence of functions bounded above by 1, $Y=\lim_{n\to\infty} Y_n$ is well defined. Yet

$$\begin{aligned}
\lim_{n\to\infty} \int (Y - Y_n)^2 d\mu &= \lim_{n\to\infty} \int \Bigl(\sum_{i=n+1}^{\infty} I(C_i)[2^i\mu(C_i) \vee 2^i]^{-1} \Bigr)^2 d\mu \\
&\leqq \lim_{n\to\infty} \int \sum_{i=n+1}^{\infty} I(C_i)[2^i\mu(C_i) \vee 2^i]^{-1} d\mu \\
&= \lim_{n\to\infty} \sum_{i=n+1}^{\infty} [2^i\mu(C_i) \vee 2^i]^{-1} \int I(C_i) d\mu \\
&\leqq \lim_{n\to\infty} \sum_{i=n+1}^{\infty} [2^i\mu(C_i)]^{-1}\mu(C_i) \\
&= \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 1/2^i = 0,
\end{aligned}$$

since one can interchange the summation sign with the integral in the case of nonnegative functions. Thus $Y \in M$ by (1). But it is easily shown that $C = \bigcup_{i=1}^{\infty} C_i = [Y > 0] \in \mathfrak{L}^+$, so that $\mathfrak{L}^+$ is closed under countable unions.

Suppose now that C is a countable intersection of sets in $\mathfrak{L}^+$. Then we may assume that $C = \bigcap_{i=1}^{\infty} C_i$ where $C_i = [X_i > 0]$, $X_i \in M$, since $[X > a] = [I([X > a]) > 0]$ where $a > 0$. Let $A_n = [X_1 > 1/n]$, so that $\bigcup_{n=1}^{\infty} A_n = C_1$. Then $[(I(A_n) + I([X_i > 1/k]) > 1] = A_n \cap [X_i > 1/k] \in \mathfrak{L}^+$ for all positive integers i, k, and n. Thus $\bigcup_{k=1}^{\infty} \{A_n \cap [X_i > 1/k]\} = A_n \cap \{\bigcup_{k=1}^{\infty} [X_i > 1/k]\} = A_n \cap C_i \in \mathfrak{L}^+$ for all i and n. By a previous argument, $Z_n = \lim_{m \to \infty} \sum_{i=2}^{m} I(A_n \cap C_i)[2^i \mu(A_n \cap C_i) \vee 2^i]^{-1} \in M$ for all n. Thus by (2), $I([Z_n \geqq \sum_{i=2}^{\infty} [2_{\mu}^{i}(A_n \cap C_i) \vee 2^i]^{-1}]) \in M$ so that

$$\left[I\left(\left[Z_n \geqq \sum_{i=2}^{\infty} [2^i \mu(A_n \cap C_i) \vee 2^i]^{-1}\right]\right) > 0\right]$$

$$= \bigcap_{i=2}^{\infty} [A_n \cap C_i] = A_n \cap \left(\bigcap_{i=2}^{\infty} C_i\right) \in \mathfrak{L}^+,$$

so that $C = \bigcap_{i=1}^{\infty} C_i = \bigcup_{n=1}^{\infty} \{A_n \cap (\bigcap_{i=2}^{\infty} C_i)\} \in \mathfrak{L}^+$.

Let us now show that $\mathfrak{L}^-$ is also closed under countable unions and countable intersections. Clearly it will suffice to show that this is true for $(\mathfrak{L}^-)^c$. But

$$(\mathfrak{L}^-)^c = \{[X < -a]; a \geqq 0, X \in M\} = \{[-X > a]; a \geqq 0, X \in M\}$$
$$= \{[Y > a]; a \geqq 0, Y \in -M\}.$$

However, this is how $\mathfrak{L}^+$ is defined with the exception that M is replaced by $-M$. Clearly, $-M$ has precisely the properties that M has, and hence $(\mathfrak{L}^-)^c$ is closed under countable unions and countable intersections by an earlier part of the theorem.

If $A \in \mathfrak{L}^+$ and $B \in \mathfrak{L}^-$ where $\mu(A) < \infty$, and $\mu(B^c) < \infty$, then $Z = I(A) - I(B^c) \in M$, so that $A \cap B = [Z > 0] \in \mathfrak{L}^+$ and $A \cup B = [Z \geqq 0] \in \mathfrak{L}^-$. In general, if $A \in \mathfrak{L}^+$, and $B \in \mathfrak{L}^-$, then there exists a sequence $\{A_i\} \in \mathfrak{L}^+$ and a sequence $\{B_i\} \in \mathfrak{L}^+$ such that $\mu(A_i) < \infty$, $\mu(B_j^c) < \infty$ for all positive integers i and j, $A = \bigcup_{i=1}^{\infty} A_i$, and $B = \bigcap_{i=1}^{\infty} B_j$. Then $A \cap B = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} (A_i \cap B_j) \in \mathfrak{L}^+$, and $A \cup B = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} (A_i \cup B_j) \in \mathfrak{L}^-$, so that $\mathfrak{L} = \mathfrak{L}^+ \cup \mathfrak{L}^-$ is a σ-lattice containing Φ and Ω. It is easily shown that $M \subset L_2(\mathfrak{L})$. The reverse inclusion can be shown by separating an arbitrary member f of $L_2(\mathfrak{L})$ into its positive and negative part, and approximating each part by the respective simple functions

$$f_n^+ = \sum_{i=1}^{n2^n} 1/2^n I[f^+ \geqq i/2^n],$$

and

$$-f_n^- = -\sum_{i=1}^{n2^n} 1/2^n I[-f^- \leqq -i/2^n],$$

which belongs to M. Then using the fact that these simple functions converge in L_2 to the respective positive and negative parts of f, and the fact that M is closed under addition, we have our desired result.

In [**1**], condition (2) can be replaced by the condition that M contain all constant functions when $\mu(\Omega) < \infty$. The corresponding condition of the revised theorem cannot be weakened in this manner.

Reference

1. H. D. Brunk, *On an extension of the concept conditional expectation*, Proc. Amer. Math. Soc. **14** (1963), 298–304. MR **26** #5599.