

# GENERALIZED THEORIES FOR $k$ -SPACES

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1. **Introduction.** Steenrod [5] has indicated that the category  $\mathcal{K}$  of  $k$ -spaces (i.e., compactly generated Hausdorff spaces with basepoints) is a convenient category for algebraic topologists for the reasons that it is large enough to contain most spaces of interest, it is closed under standard operations, and it is small enough so that these operations are nicely related. As a further argument for the convenience of the category  $\mathcal{K}$ , we show here that the generalized homology and cohomology theories introduced by G. W. Whitehead [6] for the category  $\mathcal{O}$  of finite  $CW$  complexes can be extended to theories on  $\mathcal{K}$ .

2. The category  $\mathcal{K}$ . We begin by developing several properties of the category  $\mathcal{K}$  of  $k$ -spaces with basepoints and basepoint preserving continuous maps.

2.1. An arbitrary collection of spaces  $X_\alpha \in \mathcal{K} (\alpha \in A)$  has a product  $\prod_{\alpha \in A} X_\alpha$  and a sum  $\bigvee_{\alpha \in A} X_\alpha$  in  $\mathcal{K}$  in the categorical sense [5, 4.2].

2.2. The category  $\mathcal{K}$  is closed under the operation of forming Hausdorff quotient spaces [5, 2.6].

Thus, if spaces  $X_1, \dots, X_n$  are in  $\mathcal{K}$ , so is their *reduced product*  $\bigwedge_{i=1}^n X_i = X_1 \wedge \dots \wedge X_n = \prod_{i=1}^n X_i / T(X_1, \dots, X_n)$ , where the collapsed subspace  $T(X_1, \dots, X_n)$  consists of those points of the product with at least one coordinate a basepoint. If maps  $f_i: X_i \rightarrow Y_i$  ( $i=1, \dots, n$ ) are in  $\mathcal{K}$ , so is their *reduced product*  $\bigwedge_{i=1}^n f_i = f_1 \wedge \dots \wedge f_n: \bigwedge_{i=1}^n X_i \rightarrow \bigwedge_{i=1}^n Y_i$ , the map induced by their product. Therefore each space  $X \in \mathcal{K}$  determines a reduced product functor  $X \wedge: \mathcal{K} \rightarrow \mathcal{K}$ ; the special cases in which  $X$  is the unit interval  $I$  (with basepoint 0) and the 1-sphere  $S^1 = I/(0, 1)$  are referred to as the *cone* functor  $C: \mathcal{K} \rightarrow \mathcal{K}$  and the *suspension* functor  $S: \mathcal{K} \rightarrow \mathcal{K}$ . It also follows from (2.1) and (2.2) that if a map  $f: X \rightarrow Y$  is in  $\mathcal{K}$ , then so is its *mapping sequence*

$$(2.3) \quad X \xrightarrow{f} Y \xrightarrow{i(f)} C_f$$

where the *mapping cone*  $C_f$  is the quotient space obtained from  $CX \vee Y$  by identifying  $1 \wedge x \in CX$  with  $f(x) \in Y$ , and  $i(f): Y \rightarrow C_f$  is the imbedding of  $Y$  onto the base of  $C_f$ .

There is an unrestricted exponential law in  $\mathcal{K}$  [5, 5.12] so that for each space  $X \in \mathcal{K}$  the reduced product functor  $X \wedge: \mathcal{K} \rightarrow \mathcal{K}$  has a

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right adjoint. Using this fact, together with the universal formulations of the concepts of sum and quotient map, one can prove the two results below.

2.4. For  $X \in \mathcal{K}$ , the reduced product functor  $X \wedge : \mathcal{K} \rightarrow \mathcal{K}$  preserves sums.

2.5. For  $X \in \mathcal{K}$ , the reduced product functor  $X \wedge : \mathcal{K} \rightarrow \mathcal{K}$  preserves quotient maps.

Homeomorphisms which result from an application of (2.4) will be written  $(X \vee Y) \wedge Z \xleftrightarrow{a} (X \wedge Z) \vee (Y \wedge Z)$ . It follows from (2.5) that for  $X, Y$ , and  $Z$  in  $\mathcal{K}$  the topologies on  $X \wedge Y \wedge Z$ ,  $(X \wedge Y) \wedge Z$ , and  $X \wedge (Y \wedge Z)$  are identical topologies on the set  $X \times Y \times Z / T(X, Y, Z)$ . Thus the operation of forming reduced products in  $\mathcal{K}$  is associative, as well as commutative. Subsequent homeomorphisms or isomorphisms which are consequences of this associativity (commutativity) will be denoted by  $\xleftrightarrow{a}$  ( $\xleftrightarrow{c}$ ).

A second corollary to (2.5) is that for space  $Z$  and map  $f: X \rightarrow Y$  in  $\mathcal{K}$  there is a quotient map  $(CX \vee Y) \wedge Z \rightarrow C_f \wedge Z$  whose identifications agree with those of the quotient map  $C(X \wedge Z) \vee (Y \wedge Z) \rightarrow C_f \wedge Z$  under the homeomorphisms  $(CX \vee Y) \wedge Z \xleftrightarrow{a} (CX \wedge Z) \vee (Y \wedge Z) \xleftrightarrow{c} C(X \wedge Z) \vee (Y \wedge Z)$ . A homeomorphism  $C_f \wedge Z \rightarrow C_f \wedge Z$  results; it can be used to establish the following statement.

2.6. For space  $Z$  and map  $f: X \rightarrow Y$  in  $\mathcal{K}$ , there are compatible homeomorphisms between the entries of the sequence

$$X \wedge Z \xrightarrow{f \wedge 1_Z} Y \wedge Z \xrightarrow{i(f) \wedge 1_Z} C_f \wedge Z$$

and those of the mapping sequence

$$X \wedge Z \xrightarrow{f \wedge 1_Z} Y \wedge Z \xrightarrow{i(f \wedge 1_Z)} C_f \wedge Z.$$

It is known [5, 4.3] that the topology on the product of two spaces in  $\mathcal{K}$  is the usual product topology if one of the factors is locally compact, so that the notions in  $\mathcal{K}$  of homotopy, cone, suspension, mapping cone, mapping sequence, and stable homotopy are identical with those in the larger category of topological spaces with basepoints. So the standard theorems [3, p. 462] concerning the exactness of the sequences of stable homotopy groups determined by a mapping sequence (2.3) can be used, in conjunction with (2.6), to prove this next result.

2.7. For each map  $f: X \rightarrow Y$  and pair of spaces  $W, Z$  in  $\mathcal{K}$ , the following sequences of stable homotopy groups and induced homomorphisms are exact:

$$(2.8) \quad \{W, X \wedge Z\} \xrightarrow{(f \wedge 1_Z)_*} \{W, Y \wedge Z\} \xrightarrow{(i(f) \wedge 1_Z)_*} \{W, C_f \wedge Z\}$$

$$(2.9) \quad \{W \wedge C_f, Z\} \xrightarrow{(1_W \wedge i(f))_*} \{W \wedge Y, Z\} \xrightarrow{(1_W \wedge f)_*} \{W \wedge X, Z\}.$$

3. **Homology and cohomology theories on  $\mathcal{K}$ .** We are now in position to prove that each spectrum  $\mathbf{E} = (E_k; e_k: SE_k \rightarrow E_{k+1})$  (see [6]) with spaces  $E_k \in \mathcal{K}$  ( $k \in \mathbb{Z}$ ) determines an homology theory  $(H_*(-; \mathbf{E}), \sigma_*)$  on  $\mathcal{K}$ . If  $\mathcal{A}$  denotes the category of abelian groups, then a precise statement of this result is as follows.

3.1. **THEOREM.** *Given a spectrum  $\mathbf{E}$  in  $\mathcal{K}$ , there is a sequence of covariant functors*

$$H_n(-; \mathbf{E}): \mathcal{K} \rightarrow \mathcal{A},$$

together with a sequence of natural transformations

$$\sigma_n: H_n(-; \mathbf{E}) \rightarrow H_{n+1}(S(-); \mathbf{E})$$

satisfying the following conditions:

- (1) If  $f, g$  are homotopic maps in  $\mathcal{K}$ , then  $H_n(f; \mathbf{E}) = H_n(g; \mathbf{E})$  for all  $n$ .
- (2) If  $X \in \mathcal{K}$ , then  $\sigma_n(X): H_n(X; \mathbf{E}) \approx H_{n+1}(SX; \mathbf{E})$ .
- (3) If  $f: X \rightarrow Y$  is a map in  $\mathcal{K}$ , then the sequence

$$H_n(X; \mathbf{E}) \xrightarrow{H_n(f; \mathbf{E})} H_n(Y; \mathbf{E}) \xrightarrow{H_n(i(f); \mathbf{E})} H_n(C_f; \mathbf{E})$$

is exact.

**PROOF.** For  $X$  in  $\mathcal{K}$ , we define  $H_n(X; \mathbf{E})$  to be the direct limit of the stable homotopy groups  $\{S^{k+n}, E_k \wedge X\}$  with respect to the homomorphisms

$$(e_k \wedge 1)_* \circ S_*: \{S^{k+n}, E_k \wedge X\} \rightarrow \{S^{k+1+n}, SE_k \wedge X\} \rightarrow \{S^{k+1+n}, E_{k+1} \wedge X\},$$

and we define  $\sigma_n(X): H_n(X; \mathbf{E}) \rightarrow H_{n+1}(SX; \mathbf{E})$  to be the direct limit of the system of homomorphisms

$$\{S^{k+n}, E_k \wedge X\} \xrightarrow{(-1)^k S_*} \{S^{k+1+n}, SE_k \wedge X\} \xleftarrow{c} \{S^{k+1+n}, E_k \wedge SX\},$$

where  $S_*$  is the isomorphism determined by the following correspondence of homotopy classes:

$$[S^r \wedge S^{k+n}, S^r \wedge (E_k \wedge X)] \xrightarrow{a} [S^{r-1} \wedge S^{k+1+n}, S^{r-1} \wedge (SE_k \wedge X)].$$

Once the homomorphisms  $H_n(f; \mathbf{E})$  induced by maps  $f$  in  $\mathcal{K}$  are defined in the obvious fashion, then the functorality of  $H_n$ , the naturality of  $\sigma_n$ , and the homotopy property (1) are easily checked. The isomorphism (2) follows from the fact that  $\sigma_n$  is the direct limit of isomorphisms. The exactness property (3) holds since the sequence in question is the direct limit of exact sequences of the form (2.8).

Each spectrum  $\mathbf{E}$  in  $\mathcal{K}$  determines a cohomology theory ( $H^*(-; \mathbf{E}), \sigma^*$ ) on  $\mathcal{K}$  in the following sense.

3.2. THEOREM. *Given a spectrum  $\mathbf{E}$  in  $\mathcal{K}$ , there is a sequence of contravariant functors*

$$H^n(-; \mathbf{E}): \mathcal{K} \rightarrow \mathcal{G},$$

together with a sequence of natural transformations

$$\sigma^n: H^{n+1}(S(-); \mathbf{E}) \rightarrow H^n(-; \mathbf{E})$$

satisfying the following conditions:

(1) *If  $f, g$  are homotopic maps in  $\mathcal{K}$ , then  $H^n(f; \mathbf{E}) = H^n(g; \mathbf{E})$  for all  $n$ .*

(2) *If  $X \in \mathcal{K}$ , then  $\sigma^n(X): H^{n+1}(SX; \mathbf{E}) \approx H^n(X; \mathbf{E})$ .*

(3) *If  $f: X \rightarrow Y$  is a map in  $\mathcal{K}$ , then the sequence*

$$H^n(C_f; \mathbf{E}) \xrightarrow{H^n(i(f); \mathbf{E})} H^n(Y; \mathbf{E}) \xrightarrow{H^n(f; \mathbf{E})} H^n(X; \mathbf{E})$$

is exact.

PROOF. For  $X$  in  $\mathcal{K}$ ,  $H^n(X; \mathbf{E})$  is the direct limit of the stable homotopy groups  $\{S^{k-n} \wedge X, E_k\}$  with respect to the homomorphisms

$$e_{k*} \circ S_*: \{S^{k-n} \wedge X, E_k\} \rightarrow \{S^{k+1-n} \wedge X, SE_k\} \rightarrow \{S^{k+1-n} \wedge X, E_{k+1}\},$$

and  $\sigma^n(X): H^{n+1}(SX; \mathbf{E}) \rightarrow H^n(X; \mathbf{E})$  is the direct limit of the system of homomorphisms

$$\{S^{k-(n+1)} \wedge SX, E_k\} \xrightarrow{a} \{S^{k-n} \wedge X, E_k\}.$$

When restricted to the subcategory of  $\mathcal{K}$  consisting of finite CW complexes these theories are equivalent to those presented in [6], where the spectra considered consist of CW complexes. For such spectra we have this result.

3.3. THEOREM. *If  $\mathbf{E}$  is a spectrum of CW complexes, then the homology theory  $H_*(-; \mathbf{E})$  has the property of compact support on the subcategory  $\mathfrak{W}$  of CW complexes, and the cohomology theory  $H^*(-; \mathbf{E})$  has the continuity property on the subcategory  $\mathcal{C}$  of compact Hausdorff spaces.*

PROOF. By the compact support property for  $H_*(-; \mathbf{E})$  on  $\mathfrak{W}$ , we mean that for  $X$  in  $\mathfrak{W}$ , the inclusions  $i_\alpha: X_\alpha \rightarrow X$  of the compact subspaces  $X_\alpha$  of  $X$  determine an isomorphism

$$(3.4) \quad \text{Dir Lim } H_*(i_\alpha; \mathbf{E}): \text{Dir Lim } H_*(X_\alpha; \mathbf{E}) \rightarrow H_*(X; \mathbf{E}).$$

Using the conditions (1)–(3) in (3.1), it can be shown that it is equivalent to require that (3.4) be surjective. This surjectivity is an immediate consequence of the cellular approximation theorem.

In view of the proof [1, Chapter 10, Theorem 3.1] of the continuity property of Čech cohomology, to prove the continuity of  $H^*(-; \mathbf{E})$  on  $\mathfrak{C}$  it suffices to show that it is a Čech functor in the sense that for  $Y$  in  $\mathfrak{C}$ , canonical maps  $p_u: Y \rightarrow Y_u$  associated with finite open coverings  $u \in \text{COV}(Y, \gamma_0)$  determine an isomorphism

$$\text{Dir Lim } H^*(p_u; \mathbf{E}): \text{Dir Lim } H^*(Y_u; \mathbf{E}) \rightarrow H^*(Y; \mathbf{E}).$$

That  $H^*(-; \mathbf{E})$  is a Čech functor when  $\mathbf{E}$  consists of CW complexes follows from the result that the homotopy functor  $[-; Z]$  is a Čech functor on  $\mathfrak{C}$  if  $Z$  has the homotopy type of a CW complex [4, p. 228]. The reason is that  $H^*(-; \mathbf{E})$  can be expressed as a direct limit of the homotopy functors  $[-; E_k^{S^{k-n}}]$ , where the function spaces  $E_k^{S^{k-n}}$  of base point preserving maps in the compact-open topology have the homotopy type of CW complexes by [2].

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