FATOU'S LEMMA IN SEVERAL DIMENSIONS[1]

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ABSTRACT. In this note the following generalization of Fatou's lemma is proved:

LEMMA. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of integrable functions on a measure space S with values in R_+^d, the nonnegative orthant of a d-dimensional Euclidean space, for which $\int f_n \to a \in R_+^d$. Then there exists an integrable function f, from S to R_+^d, such that a.e. $f(s)$ is a limit point of $(f_n(s))_{n=1}^{\infty}$ and $\int f \leqq a$.*

1. **Introduction.** When $d=1$, the result is a form of Fatou's lemma. The assertion above is applied in mathematical economics [4].[2] It is also strongly connected with the theory of set valued functions [2] or correspondences [3]. The nontrivial part of the arguments is limited to the case where S is an atomless measure space. In the purely atomic case the lemma is reduced to a simple exercise in series. In any case, the lemma cannot be proved by a successive application of Fatou's lemma d times.

A few corollaries of the lemma are proved in §3.

2. **Preliminary results and the proof of the lemma.** Let $(A_n)_{n=1}^{\infty}$ be a sequence of (nonempty) subsets of R^d. We denote by Lim $\text{Sup}_n A_n$ the set of all the limit points of the sequences $(a_n)_{n=1}^{\infty}$ with $a^n \in A_n$, $n=1, 2, \cdots$. Denote by $x \cdot y$ the inner product, $\sum_{i=1}^{d} x^i y^i$, in R^d.

PROPOSITION 1. *For each $p \gg 0$ there is an integrable function g such that $p \cdot \int g \leqq p \cdot a$ and a.e. $g(s) \in \text{Lim Sup}_n \{f_n(s)\}$ and $p \cdot g(s) = \lim \inf_n p \cdot f_n(s)$.*

PROOF. Define $h(s) = \lim \inf_n p \cdot f_n(s)$. As $p \cdot f_n(s) \to p \cdot a$, by Fatou's lemma $\int h \leqq p \cdot a$. Now we decompose h to d integrable components $g^1, \cdots, g^d$ such that a.e. $p \cdot g(s) = h(s)$.

Define:

$$g_n^1(s) = \inf\{f_k^1(s) \mid k \geqq n \quad \text{and} \quad p \cdot f_k(s) < h(s) + 1/n\}.$$

For each $r \in R^1$ and $n = 1, 2, \cdots$ one has

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$$\{s \mid g_n^1(s) < r\} = \bigcup_{k \geqq n} (\{s \mid f_k^1(s) < r\} \cap \{s \mid p \cdot f_k(s) < h(s) + 1/n\}).$$

Hence (g_n^1) is a monotone sequence of measurable functions, each bounded by the integrable function $(1/p^1)(h+1)$. Define $g^1(s) = \lim_n g_n^1(s)$ then g^1 is an integrable function and a.e. $p^1 g^1(s) \leqq h(s)$ and $g^1(s) \in \text{Lim Sup}_n \{f_n^1(s)\}$.

Proceed by induction:

$$g_n^i(s) = \inf\{f_k^i(s) \mid k \geqq n \text{ and } p \cdot f_k(s) < h(s) + 1/n \text{ and } f_k^j(s) < g^j(s) + 1/n, j = 1, \cdots, i-1\}.$$

It is easy to check that $g_n^i(s)$ is well defined and $g^i(s) = \lim_n g_n^i(s)$ is an integrable function with $\sum_{j=1}^i p^j g^j(s) \leqq h(s)$ and $(g^1(s), \cdots, g^i(s)) \in \text{Lim Sup}_n \{(f_n^1(s), \cdots, f_n^i(s))\}$ a.e. After d steps we have $g(s) = (g^1(s), \cdots, g^d(s))$ such that a.e. $p \cdot g(s) = h(s)$ and $g(s) \in \text{Lim Sup}_n \{f_n(s)\}$. Q.E.D.

Denote: $Q_y = \{x \in R_+^d \mid x \leqq y\}$, $y \in R_+^d$.

PROPOSITION 2. *Let A be a closed, convex subset of R_+^d and $y \in R_+^d$ such that $A \cap Q_y = \varnothing$. Then there is $q \gg 0$ with*

$$\sup\{q \cdot x \mid x \in Q_y\} < \inf\{q \cdot x \mid x \in A\}.$$

PROOF. By the separation theorem there are p and α such that $p \cdot x < \alpha < p \cdot z$ for all $x \in Q_y$ and all $z \in A$. Let p' be the vector obtained from p by substitution of zero for each negative coordinate of p. For $x \in A$, $x \geqq 0$ so $p' \cdot x \geqq p \cdot x > \alpha$. For $x \in Q_y$ let x' denote the vector obtained from x by substitution of zero for those coordinates which we changed previously in p. Of course, $x' \in Q_y$, so $p' \cdot x = p \cdot x' < \alpha$. Denote by p_δ the vector obtained from p' by substitution of $\delta > 0$ for each zero in p'. Again for each $x \in A$ $p_\delta \cdot x \geqq p' \cdot x > \alpha$. For $x \in Q_y$ one has $p_\delta \cdot x \leqq p' \cdot x + d\delta < \alpha + d\delta$. Because of the compactness of Q_y there is a $\delta' > 0$ such that $q = p_{\delta'}$ fulfills the requirements of the proposition. Q.E.D.

For each s in S let $F(s)$ be a nonempty subset of R^d.

Following [2] we define:

$$\int F = \left\{ \int h \,\middle|\, h \text{ is integrable and a.e. } h(s) \in F(s) \right\}$$

PROPOSITION 3. *Let $A = \int \text{Lim Sup}_n \{f_n(s)\}$ and $q \gg 0$ such that for each $x \in A$, $q \cdot x \geqq q \cdot a$. Then there is a subsequence $(f_{n_k})_{k=1}^\infty$ of (f_n) such that for each $x \in \int \text{Lim Sup}_k \{f_{n_k}(s)\}$, $q \cdot x = q \cdot a$.*

PROOF. Denote $h_n(s)=\inf\{q\cdot f_k(s)\mid k\geqq n\}$ and $h(s)=\lim_n h_n(s)$ $=\liminf_n q\cdot f_n(s)$. Using Proposition 1 for $p=q$ one has: $\int h=q\cdot\int g$ $\leqq q\cdot a$ and $\int g\in A$. By the condition of the proposition $q\cdot\int g\geqq q\cdot a$; so $\int h=q\cdot a$. For each $s\in S$, $h_n(s)\leqq q\cdot f_n(s)$ so $\int|q\cdot f_n-h_n|=\int(q\cdot f_n-h_n)$ $=\int q\cdot f_n-\int h_n\to q\cdot a-\int h=0$. Also as $\int|h-h_n|\to 0$, we get $\int|q\cdot f_n-h|$ $\to 0$. Convergence in the mean implies the convergence of a subsequence a.e. Hence there is a subsequence (f_{n_k}) such that a.e. $q\cdot f_{n_m}(s)$ $\to h(s)$. Consequently for a.e. $s\in S$ and each $x\in\text{Lim Sup}_k\{f_{n_k}(s)\}$, $q\cdot x=h(s)$. Integrating over S completes the proof. Q.E.D.

PROPOSITION 4. *Let S be atomless and for each $s\in S$ let $F(s)$ be a nonempty subset of R^d. Then $\int F$ is convex.*

PROOF. This is an elementary theorem about integrals of correspondences due to Richter, [**5**]. (The proof appears also in [**3**], p. 369].) This theorem is a simple consequence of Lyapunov convexity theorem and will not be reproved here.

PROPOSITION 5. *Let $a_{k,n}\in R^d_+$ for $k,\ n=1,2,\cdots$ and assume that $\sum_{k=1}^\infty a_{k,n}\to a$ (where $n\to\infty$). Then there is a sequence $(b_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty b_k\leqq a$ and for each k, $b_k\in\text{Lim Sup}_n\{a_{n,k}\}$. Moreover, if there is in addition, a sequence $(c_k)_{k=1}^\infty$ such that for each n and k, $a_{n,k}\leqq c_k$ and $\sum_{k=1}^\infty c_k=c\in R^d_+$, then $\sum_{k=1}^\infty b_k=a$.*

REMARK. The first part of this proposition is exactly the statement of the lemma in case of a purely atomic measure space; the second part is related to Corollary 1 in §3.

PROOF. Reasoning by compactness, the sequence of sequences $((a_{k,n})_{k=1}^\infty)_{n=1}^\infty$ has a pointwise converging subsequence $((a_{k,n_j})_{k=1}^\infty)_{j=1}^\infty$, the limit of which we denote by $(b_k)_{k=1}^\infty$. Thus, for each k, $b_k=\lim_j a_{k,n_j}$ i.e. $b_k\in\text{Lim Sup}_n\{a_{k,n}\}$. We have to prove that $\sum_{k=1}^\infty b_k\leqq a$. Assume the contrary, i.e. there is a coordinate i, an integer N and a number $\epsilon>0$ such that $\sum_{k=1}^N b_k^i\geqq a^i+\epsilon$. For each k let M_k be such that $n_j>M_k$ imply $b_k^i<a_{k,n_j}^i+\epsilon/2N$, and let M_0 be such that $n>M_0$ imply $\sum_{k=1}^\infty a_{k,n}^i$ $<a^i+\epsilon/4$. Define $M=\max\{M_0,M_1,\cdots,M_N\}$. Then for $n_m>M$ one has: $\sum_{k=1}^\infty a_{k,n_m}^i\geqq\sum_{k=1}^N a_{k,n_m}^i>\sum_{k=1}^N b_k^i-\epsilon/2\geqq a^i+\epsilon/2$, a contradiction.

Now assume the additional condition and apply the first part of the proposition to the sequence $((c_k-a_{k,n_j})_{k=1}^\infty)_{j=1}^\infty$. Q.E.D.

A point x in a set B in R^d is called *admissible* if $x\geqq y\in B$ imply $x=y$. If for $x\in B$ there exists a vector $p\gg 0$ such that for each $y\in B$, $p\cdot x\leqq p\cdot y$ then x is called *strictly admissible*. Of course, a strictly admissible point of a set is also admissible.

PROPOSITION 6. *The admissible points of a closed convex set in R^d belong to the closure of the strictly admissible points of this set.*

PROOF. This is a theorem of Arrow-Barankin-Blackwell, [1]. (I thank G. Debreu for this reference.)

PROPOSITION 7. *Let S be atomless and set $A = \int \text{Lim Sup}_n \{f_n(s)\}$. Then A is convex and $\bar{A} \cap Q_a \neq \emptyset$.*

PROOF. The convexity of A is implied by Proposition 4. A is nonempty by Proposition 1. Assume that $\bar{A} \cap Q_a = \emptyset$. By Proposition 2 there is a vector $q \gg 0$ with

$$\inf\{q\cdot x \mid x \in A\} > q\cdot a \; (q\cdot a = \sup\{q\cdot x \mid x \in Q_a\}).$$

The last inequality contradicts Proposition 1. Q.E.D.

PROOF OF THE LEMMA. We decompose S to an atomless part and a purely atomic part. The lemma can be proved separately for each part. Proposition 5, as remarked above, proves the lemma for the purely atomic case. (One can assume, without a loss of generality, that in S there are at most $\aleph_0$ atoms.)

Now assume that S is atomless. We prove the lemma reasoning by induction on dim (A). (A denotes the $\int \text{Lim Sup}_n \{f_n(s)\}$ and dim (A) is the linear dimension of the smallest flat containing A.) By Proposition 7, dim $(A) \geqq 0$ and if dim $(A) = 0$ then the lemma holds. Given $0 < l \leqq d$ assume that the lemma holds when dim $(A) < l$ and we shall prove it for the case dim $(A) = l$. The induction hypothesis states that for each sequence of integrable functions $g_n: S \to R^d_+$ with $\int g_n \to b$ and dim $(\int \text{Lim Sup}_n \{g_n(s)\}) < l$ one has

$$\int \operatorname*{Lim\,Sup}_n \{g_n(s)\} \cap Q_b \neq \emptyset.$$

In view of Proposition 7 it is sufficient to prove that the admissible points of $\bar{A}$ belong to A.

Claim 1. The strictly admissible points of $\bar{A}$ belong to A.

Let $b \in \bar{A}$ and $q \gg 0$ such that $q\cdot b \geqq q\cdot x$ for each $x \in \bar{A}$. If $b \in$ rel-int A then $b \in A$ because of the convexity of A. In the other case the origin is a boundary point of $A - \{b\}$ in the subspace $H - \{b\}$ of R^d, where H is the smallest flat containing A. Then there is $q' \neq 0$ in $H - \{b\}$ for which $q'\cdot x \geqq 0$ for each $x \in A - \{b\}$ and for at least one point of $A - \{b\}$, say x_0, $q'\cdot x_0 > 0$. Hence there is $\epsilon > 0$ such that defining $p = \epsilon q' + (1-\epsilon)q$ we have: $p \gg 0$, $\forall x \in A$, $p\cdot x \geqq p\cdot b$ and a strict inequality for at least one point of A. So

$$\dim(A \cap \{x \mid p\cdot x = p\cdot b\}) < l.$$

Let $y_n \to b$ with $y_n \in A$ for $n=1, 2, \cdots$. Hence there is a sequence of integrable functions (g_n) such that for each n, $\int g_n = y_n$ and a.e. $g_n(s) \in \text{Lim Sup}_k \{ f_k(s) \}$. Define $B = \int \text{Lim Sup}_n \{ g_n(s) \}$, then $B \subset A$ because a.e. $\text{Lim Sup}_n \{ g_n(s) \} \subset \text{Lim Sup}_n \{ f_n(s) \}$. In consequence $p \cdot x \geqq p \cdot b$ for each $x \in B$ and $\dim(B \cap \{ x \mid p \cdot x = p \cdot b \}) < l$. Now the conditions of Proposition 3 are fulfilled for (g_n), B, b and p, hence there is a subsequence $(g_{n_k})_{k=1}^{\infty}$ such that

$$\int \operatorname*{Lim\,Sup}_k \{ g_{n_k}(s) \} \subset B \cap \{ x \mid p \cdot x = p \cdot b \}.$$

The induction hypothesis completes the proof of the claim.

Claim 2. The admissible points of $\bar{A}$ belong to A.

Denote by b an admissible point of $\bar{A}$. Because of Proposition 6 and Claim 1 there is a sequence (y_n) of strictly admissible points in A such that $y_n \to b$. Set (g_n), B and H as in the proof of Claim 1. For each n there is a vector q_n with $q_n \cdot q_n = 1$ and $q_n \cdot x \geqq q_n \cdot y_n$ for each $x \in A$. We may assume, in addition that for each n, $q_n \in H - \{ y_n \}$. (Note that $H - \{ y_n \} = H - \{ b \}$ for each n.) Otherwise, we have for some n, that 0 is an interior point of $A - \{ y_n \}$ in $H - \{ y_n \}$, or equivalently: $y_n \in$ rel-int A. But then, because y_n is a strictly admissible point of $\bar{A}$, it implies that each point of $\bar{A}$ is strictly admissible and in this case Claim 2 is a consequence of Claim 1. Thus (q_n) has a limit point q in $H - \{ b \}$. Assume, without loss of generality, that $q_n \to q$. For each $x \in A - \{ b \}$, $q \cdot x \geqq 0$ and $\dim (\{ x \in H - (b) \mid q \cdot x = 0 \}) < l$. Hence, in order to complete the proof of the claim by the induction hypothesis, it is sufficient to show that for each $x \in B$, $q \cdot x = q \cdot b$.

Assume, per absurdum, that there is $x_0 \in B$ with $q \cdot x_0 > q \cdot b$. Because $b \in \bar{A}$ there is $z \in B$ with $q \cdot x_0 > q \cdot z$. Let h and g be two integrable functions such that: $x_0 = \int g$, $z = \int h$ and a.e. $g(s) \in \text{Lim Sup}_n \{ g_n(s) \}$ and $h(s) \in \text{Lim Sup}_n \{ f_n(s) \}$. As a consequence of the last inequality there is a nonnull set U defined:

$$U = \{ s \in S \mid q \cdot h(s) < q \cdot g(s) \}.$$

Consequently, for each $s \in U$ there are $N(s)$ and $\epsilon(s) > 0$ such that for each $n > N(s)$, $q_n \cdot h(s) < q_n \cdot g(s) + \epsilon(s)$. Because $g(s) \in \text{Lim Sup}_n \{ g_n(s) \}$ there is $n(s) > N(s)$, $s \in U$, such that $q_{n(s)} \cdot h(s) < q_{n(s)} \cdot g_{n(s)}(s)$. Since $U = \bigcup_{k=1}^{\infty} \{ s \in U \mid n(s) = k \}$, there are k and a nonnull subset, V, of U such that for each $s \in V$, $q_k \cdot h(s) < q_k \cdot g_k(s)$. Define a function $\hat{g}$ by: $\hat{g}(s) = h(s)$ for $s \in V$ and $\hat{g}(s) = g_k(s)$ for $s \notin V$. Then a.e. $\hat{g}(s) \in$ Lim $\text{Sup}_n \{ f_n(s) \}$ and $q_k \cdot \int \hat{g} < q_k \cdot \int g_k = q_K \cdot y_K$—a contradiction. Q.E.D.

3. **Corollaries.** The first two corollaries were proved by Aumann [2], (with some restrictions on S). Our proof, based on the lemma, is shorter and simpler than his direct proof. These corollaries have direct application in mathematical economy [4], [6]. As to Corollary 4, it is natural to assume that it has a direct elementary proof but not as short as the one below.

COROLLARY 1. *Let (f_n) be a sequence of integrable functions from S to R^d such that $\int f_n \to a$ and there is an integrable function g with $|f_n(s)| \leqq |g(s)|$ for a.e. $s \in S$ and $n = 1, 2, \cdots$. Then there is an integrable function f such that $\int f = a$ and a.e. $f(s)$ is a limit point of $(f_n(s))$.*

PROOF. As in the proof of the lemma, we can deal separately with each of the two cases: S is atomless, S is purely atomic. In the second case, Proposition 5 proves the corollary. We assume, for the rest of the proof, that S is atomless.

Let d_i; $i = 1, \cdots, 2^d$ be the vectors in R^d with coordinates 1 or -1. Define $d_i \nabla x$ to be the vector in R^d the jth coordinate of which is $d_i^j x^j$ and define $e = (1, \cdots, 1)$. Then for each n and a.e. s, $d_i \nabla f_n(s) \leqq e|g(s)|$, $i = 1, \cdots, 2^d$. Now apply the lemma to $(e|g| + d_i \nabla f_n)_{n=1}^{\infty}$, $i = 1, \cdots, 2^d$. For each i we get an integrable function h_i such that $d_i \nabla \int h_i \leqq d_i \nabla a$ and a.e. $h_i(s) \in \text{Lim Sup}_n \{f_n(s)\}$. So, using Proposition 4, we get: $a \in \int \text{Lim Sup}_n \{f_n(s)\}$. Q.E.D.

COROLLARY 2. *For each s let $(F_n(s))$ be a sequence of nonempty subsets of R^d with the property: $x \in F_n(s)$ imply $|x| \leqq g(s)|$, for some integrable function g. Then*

$$\operatorname*{Lim\,Sup}_n \int F_n \subset \int \operatorname*{Lim\,Sup}_n F_n(s).$$

PROOF. Assume that $x \in \text{Lim Sup}_n \int F_n$. Then x is a limit point of a sequence (x_n) with $x_n \in \int F_n$. To simplicate notation assume that $x_n \to x$. $x_n \in \int F_n$ means that $x_n = \int f_n$ for an integrable function f_n with $f_n(s) \in F_n(s)$ a.e. By Corollary 1 there is an integrable function f with $\int f = x$ and a.e. $f(s) \in \text{Lim Sup}_n \{f_n(s)\}$. Hence we completed the proof since a.e. $\text{Lim Sup}_n \{f_n(s)\} \subset \text{Lim Sup}_n F_n(s)$. Q.E.D.

COROLLARY 3. *Let F be a closed-valued correspondence from S to R_+^d i.e. $F(s)$ is a nonempty, closed subset of R_+^d, for each $s \in S$. Then $\int F$ contains all the admissible points of its closure.*

PROOF. Let x be an admissible point of $\int F$. Then there is a sequence $x_n \to x$ with $x_n \in \int F$ for each n. It means that there is a sequence (f_n) of integrable functions with $f_n(s) \in F(s)$ a.e. for $n = 1, 2, \cdots$. By the lemma there is an integrable function f with $\int f \leqq x$ and a.e. $f(s) \in \text{Lim Sup}_n \{f_n(s)\} \subset F(s)$, where the inclusion is implied by the condition that $F(s)$ is closed for each s. Hence $\int f \in \int F$ and because x is an admissible point, we get $\int f = x \in \int F$. Q.E.D.

COROLLARY 4. *Let A be a closed set in R^d_+. Then* conv (A) *contains all the admissible points of its closure.*

PROOF. Let S be an atomless probability measure space. (The word "Probability" means that the measure of S is 1.) Define $F(s) = A$ for each $s \in S$. Then, by Corollary 3, the following claim completes the proof.

Claim 3. Let S be an atomless probability measure space and A in R^d. Then conv $(A) = \int F$, where $F(s) = A$ for each $s \in S$.

By Proposition 4, conv $(A) \subset \int F$. The other inclusion can be easily proved by induction of the dimension and is left to the reader. Q.E.D.

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