A SEQUENTIALLY CLOSED COUNTABLE DENSE SUBSET OF I^I

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Let I denote the closed unit interval, N the positive integers, and R the reals. By I^I we mean the collection of all functions $x=x(t)$ from I into I with the product topology. A sequentially closed set in I^I, i.e., a set containing all its sequential limits, is not in general closed, I^I not being first countable. In fact, there are proper subsets of I^I that are sequentially closed and dense, e.g.,

$$\{x \in I^I \mid x \text{ is Lebesgue measurable}\}$$

and

$$\{x \in I^I \mid x^{-1}(I\backslash\{0\}) \text{ is countable}\}.$$

These sets have cardinality 2^c and c, respectively. In [5], we raised the question of the existence of a *countable* such set. We here answer this question affirmatively by showing that I^I contains a countable dense subset that contains no nontrivial convergent sequences at all. Separability of I^I is well known, and follows, for instance, from a more general result of Pondiczery [4] (see also Marczewski [3]).

Let $J\subset R$ be a maximal subset of irrationals linearly independent over the integers. Since J and I have the same cardinality, I^I and I^J are homeomorphic, and so our goal will be reached by exhibiting a countable dense subset of I^J having no nontrivial convergent sequences in it. Let S be the set $\{x_n\}_{n\in N}$ in I^J such that

$$x_n(t) = (nt) \quad \text{for all } t \in J,$$

where (r), for $r\in R$, denotes the number in $[0, 1)$ congruent to r modulo 1. The fact that S is dense in I^J follows directly from Kronecker's well-known theorem regarding the simultaneous approximation mod 1 of arbitrary real numbers by integral multiples of linearly independent irrationals [2, Chapter XXIII].

Now consider any convergent sequence $\{x_{n_k}\}$ of elements of S. Then $\{x_{n_k}(t)\}$ converges for each $t\in J$. For $t\in R$, there exist, by the maximality of J, finitely many integers m_i such that $m_0t =m_1 1+\sum_{i\geq 2} m_it_i$ for some $t_i\in J$. Therefore $(m_0n_kt)=(\sum_{i\geq 2} m_ix_{n_k}(t_i))$, implying that [if we agree to identify 0 and 1] the sequence $\{(m_0n_kt)\}$ converges as $k\to\infty$. This in turn implies that $\{(n_kt)\}_{k\in N}$ has only

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finitely many [in fact, at most $m_0 = m_0(t)$] limit points in I for each $t \in R$. This is impossible unless $\{n_k\}_{k\in N}$ is finite, since otherwise $\{(n_k t)\}_{k \in N}$ is dense in I for almost all $t \in R$ by a theorem of Hardy and Littlewood [1, Theorem 1.40]. That completes the proof.

The index set J in the above proof may be of measure zero, since $J \cup \{1\}$ is a Hamel basis of the reals over the rationals, and there exist Hamel bases of measure zero [6]. For this reason the Hardy-Littlewood theorem could not be applied directly to our situation. We could have avoided this slight holdup by initially choosing a nonmeasurable J. Such a J exists, since there exists a Hamel basis which is not a Lebesgue measurable set [6]. The only measurable Hamel bases have measure zero [6].

References

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