INTERSECTIONS OF MAXIMAL L_n SETS

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0. **Introduction.** In a paper by Hare and Kenelly [2], it is shown that the intersection of the maximal starshaped subsets of a compact, simply-connected set in E_2 is starshaped or empty. In this paper an investigation is made of the problem of describing the intersection of all maximal L_n subsets of a set. It will be shown that every set in E_m has maximal L_n subsets. Furthermore, if S is a compact, simply-connected set in E_2, then the intersection of the maximal L_n subsets of S is an L_n set.

1. **Preliminaries.** In the sequel, if B is a set, then B^c will denote its complement, $\bar{B}$ its closure, and bd B its boundary. If x and y are points, then $P_n(x, y)$ will denote a polygonal n-path joining x to y. The notation $[p_0, p_1, p_2, \cdots, p_m]$ will denote a polygonal m-path joining p_0 to p_m and having $p_1, p_2, \cdots, p_{m-1}$ as consecutive, intermediate vertices. Variations of this notation, analogous to those customary for a segment on the real line, will be used to denote the exclusion of p_0 and/or p_m. Finally if A and B are sets, then $A-B$ will denote the set $A \cap B^c$.

Let S be a set in E_m and let x be in S. Then $K(n, x, S)$ will denote the nth order kernel of x in S. If no confusion can arise, $K(n, x)$ will be used in place of $K(n, x, S)$.

DEFINITION. A compact set S in E_2 is said to be simply-connected if and only if S^c is connected.

It is most important to keep in mind that if J is a closed Jordan curve in the compact, simply-connected set S, then the interior of J is contained in S, where the interior of J is in the sense of the Jordan Curve Theorem.

Two results due to Bruckner and Bruckner [1] will be used and will be stated here for reference. For the proofs, the reader should consult the above paper. The following theorems hold for a compact, simply-connected set S in E_2:

THEOREM 1.1. *Let $x \in S$ and $y, z \in K(n, x, S)$. If $[y, z] \subset S$, then $[y, z] \subset K(n, x, S)$.*

THEOREM 1.2. *If $y \in K(n, x, S)$ for all $x, y \in$ bd S, then S is an L_n set.*

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2. The existence of maximal L_n sets.

THEOREM 2.1. *Let A be a bounded subset of E_m. If A is an L_n set, then $\overline{A}$ is an L_n set.*

PROOF. The proof is seen by a standard sequence argument using the compactness of $\overline{A}$.

THEOREM 2.2. *Let S be a set in E_m. Then there exists a subset of S which is a maximal L_n subset of S.*

PROOF. Without loss of generality, assume that $S \neq \emptyset$. Let $\mathfrak{A}$ be defined by

$$\mathfrak{A} = \{A \mid A \subset S \text{ and } A \text{ is an } L_n \text{ set}\}.$$

Then $\mathfrak{A}$ is nonvoid. For let $x \in S$, then $\{x\} \in \mathfrak{A}$.

Partially order $\mathfrak{A}$ by set inclusion, and let $\mathfrak{C}$ be a chain in $\mathfrak{A}$. It is easily seen that $\bigcup \mathfrak{C} \in \mathfrak{A}$. Thus $\bigcup \mathfrak{C}$ is an upper bound of $\mathfrak{C}$ in $\mathfrak{A}$. Hence, by Zorn's Lemma, $\mathfrak{A}$ has a maximal element.

THEOREM 2.3. *Let S be a compact set in E_m. Then each maximal L_n subset of S is closed.*

PROOF. Suppose that A is a maximal L_n subset of S. By Theorem 2.1. $\overline{A}$ is an L_n set. Since S is compact, it is clear that $\overline{A} \subset S$. Thus, $A \subset \overline{A}$, where $\overline{A}$ is an L_n subset of S. Since A is maximal, it follows that $A = \overline{A}$. Hence, A is closed.

THEOREM 2.4. *Let S be a set in E_m and let A be an L_n subset of S. Then A is contained in a maximal L_n subset of S.*

PROOF. Define the set $\mathfrak{A}'$ by

$$\mathfrak{A}' = \{B \mid A \subset B \subset S \text{ and } B \text{ is an } L_n \text{ set}\}.$$

Then $\mathfrak{A}'$ is nonvoid since $A \in \mathfrak{A}'$. Now partially order $\mathfrak{A}'$ by set inclusion.

As in the proof of Theorem 2.2, it is seen that $\mathfrak{A}'$ has a maximal element M'. Note that M' is also a maximal element of $\mathfrak{A}$, where $\mathfrak{A}$ is as in Theorem 2.2. For suppose that $M \in \mathfrak{A}$ is such that $M' \subset M$. Since M is an L_n subset of S, it follows that $M \in \mathfrak{A}'$. However, M' is maximal in $\mathfrak{A}'$ and thus $M' = M$. Thus, it is seen that M' is a maximal element of $\mathfrak{A}$ such that $A \subset M'$. This completes the proof.

3. An extension of a theorem on the generalized convex kernel.

Let S be a compact, simply-connected set in E_2. For any polygonal path $C(p, q)$ from p to q in S, let $\rho[C(p, q)]$ denote the length of the path.

DEFINITION. Suppose that $p, q \in S$ and $C(p, q)$ is a polygonal path from p to q in S. Then $C(p, q)$ is called a *minimal l-path if* $C(p, q) = [p, q]$. Let $k > 1$, then $C(p, q)$ is called a *minimal k-path* if

(a) $C(p, q)$ is a k-path,

(b) $p \notin K(k-1, q, S)$,

(c) if $C'(p, q)$ is any other k-path from p to q in S, then $\rho[C(p, q)] \leqq \rho[C'(p, q)]$.

THEOREM 3.1. *Suppose $p \in K(m, q, S)$ for some m. Then there exists a minimal k-path from p to q in S for some k such that $1 \leqq k \leqq m$.*

PROOF. If $[p, q] \subset S$, then there is nothing further to show. Hence, suppose that $[p, q] \not\subset S$. Then there exists k such that $p \in K(k, q, S)$ but $p \notin K(k-1, q, S)$. It is clear that $k \leqq m$.

Let $\mathcal{C}_k(p, q)$ denote the set of all k-paths from p to q in S. Now suppose that there does not exist a minimal k-path from p to q in S. Then there exists a sequence $\{C_j\}$ in $\mathcal{C}_k(p, q)$ such that:

1. $\rho(C_j) > \rho(C_{j+1})$ for $j = 1, 2, \cdots$;

2. for every $C \in \mathcal{C}_k(p, q)$, there exists some j_0 such that $\rho(C_{j_0}) < \rho(C)$.

For j arbitrary, let $C_j = [p, x_{1j}, x_{2j}, \cdots, x_{(k-1)j}, q]$. Since S is compact, it can be assumed without loss of generality that $\{x_{ij}\} \to x_i \in S$ for $i = 1, 2, \cdots, k-1$. Let $C_0 = [p, x_1, x_2, \cdots, x_{k-1}, q]$. Then it is easily seen that $C_0 \in \mathcal{C}_k(p, q)$ and that $\rho(C_0) < \rho(C_j)$ for every j. However, this contradicts (2).

Hence, it now follows that there exists a minimal k-path from p to q in S.

THEOREM 3.2. *Suppose that $p, q \in K(n, x, S)$. Let $C_k(p, q)$ be a minimal k-path from p to q in S, then $C_k(p, q) \subset K(n, x, S)$.*

PROOF. If $k = 1$, then the result follows from Theorem 1.1.

Hence, suppose that the theorem is true for $k \leqq m$. Now the case where $k = m+1$ will be considered. Thus, suppose that $p \in K(m+1, q)$ but $p \notin K(m, q)$. Let $C_{m+1}(p, q) = [p, x_1, x_2, \cdots, x_m, q]$ be a minimal $(m+1)$-path joining p to q in S.

If it can be shown that, in every possible case, some $x_j \in K(n, x)$ then the result will follow for $C_{m+1}(p, q)$. For, if $x_j \in K(n, x)$, then

$$\begin{aligned} C_{m+1}(p, q) &= [p, x_1, \cdots, x_j, \cdots, x_m, q] \\ &= [p, x_1, \cdots, x_j] \cup [x_j, \cdots, x_m, q] \\ &= C_j(p, x_j) \cup C_{m-j+1}(x_j, q) \end{aligned}$$

where $p, x_j, q \in K(n, x)$, $C_j(p, x_j)$ is a minimal j-path from p to x_j in S, and $C_{m-j+1}(x_j, q)$ is a minimal $(m-j+1)$-path from x_j to q in S.

Since $1 \leqq j \leqq m$, it is easy to see that $1 \leqq m-j+1 \leqq m$. Thus, by the induction hypothesis, it follows that $C_j(p,x_j) \subset K(n,x)$ and $C_{m-j+1}(x_j, q) \subset K(n, x)$. Thus, $C_{m+1}(p, q) \subset K(n, x)$.

It will now be shown that, in each possible case, there exists some $x_j \in K(n, x)$. Now, $p, q \in K(n, x)$ implies that there exist finite sequences $\{p_i\}_{i=1}^{n-1}$ and $\{q_i\}_{i=1}^{n-1}$ in S such that $[x, p_1, \cdots, p_{n-1}, p] \subset S$ and $[x, q_1, \cdots, q_{n-1}, q] \subset S$.

Case 1. If $[x, p_1, p_2, \cdots, p_{n-1}] \cap C_{m+1}(p, q) \neq \varnothing$, then it is clear that there exists $j \in \{1, 2, \cdots, m\}$ such that $x_j \in K(n, x)$.

Case 1′. If $[x, q_1, q_2, \cdots, q_{n-1}] \cap C_{m+1}(p, q) \neq \varnothing$, then it is clear that there exists $j \in \{1, 2, \cdots, m\}$ such that $x_j \in K(n, x)$.

Case 2. If $[p_{n-1}, p] \cap [p, x_1] \supsetneq \{p\}$, then $x_1 \in K(n, x)$.

Case 2′. If $[q_{n-1}, q] \cap [x_m, q] \supsetneq \{q\}$, then $x_m \in K(n, x)$.

Case 3. If $[p_{n-1}, p] \cap (x_1, x_2, \cdots, x_m, q] \neq \varnothing$, then either $p \in K(m, q)$ or there exists $C \in \mathcal{C}_{m+1}(p, q)$ such that $\rho(C) < \rho[C_{m+1}(p, q)]$. However, neither of these is possible and therefore $[p_{n-1}, p] \cap (x_1, x_2, \cdots, x_m, q] = \varnothing$.

Case 3′. Similarly, it is seen that $[q_{n-1}, q] \cap [p, x_1, x_2, \cdots, x_m) = \varnothing$.

Case 4. The only remaining possibility is when

$$[x, p_1, p_2, \cdots, p_{n-1}, p] \cap C_{m+1}(p, q) = \{p\}$$

and

$$[x, q_1, q_2, \cdots, q_{n-1}, q] \cap C_{m+1}(p, q) = \{q\}.$$

Notice that in this event there is a domain $D \subset S$ determined by $C_{m+1}(p, q) \cup [x, p_1, \cdots, p_{n-1}, p] \cup [x, q_1, \cdots, q_{n-1}, q]$ such that $C_{m+1}(p, q) \subset \text{bd } D$ and bd D is a closed Jordan curve. Actually, bd D is a polygonal curve. Let the angle having vertex x_j, sides along bd D, and interior to bd D be denoted by intan x_j. Due to the minimality of $C_{m+1}(p, q)$, it is necessary that intan $x_j > 180°$ for $j = 1, 2, \cdots, m$. Now, from x_1, extend $[p, x_1]$. Then there exists $p' \in S$ such that:

1. $p' \in (p, p_{n-1}, \cdots, p_1, x, q_1, \cdots, q_{n-1}, q)$;
2. $[p, p'] \subset S$;
3. $x_1 \in (p, p')$.

From this it is clear that $p' \in K(n, x)$. Since $p \in K(n, x)$ and $[p, p'] \subset S$, it follows from the inductive hypothesis that $[p, p'] \subset K(n, x)$. In particular, $x_1 \in K(n, x)$.

Thus, it has been shown that in every possible case, there exists $j \in \{1, 2, \cdots, m\}$ such that $x_j \in K(n, x)$. It now follows that $C_{m+1}(p, q) \subset K(n, x)$. Therefore, the theorem is true for $k = m+1$ and the induction is complete. This completes the proof.

4. Intersections of maximal L_n sets. Let S be a compact, simply-connected set in E_2. Let $\mathfrak{L}_n = \{L_\alpha | \alpha \in \Delta_n\}$ be the set of all maximal L_n subsets of S and let $A_n = \bigcap \mathfrak{L}_n$.

THEOREM 4.1. *Let G be a bounded set in E_2, then there is a smallest compact simply-connected set in E_2 which contains G.*

PROOF. Let $\mathfrak{A} = \{A_\alpha | \alpha \in \Delta\}$ be the family of all compact simply-connected sets in E_2 which contain G. Since G is bounded, it is clear that $\mathfrak{A}$ is nonvoid. Let $A = \bigcap \mathfrak{A}$, then it is clear that A is the desired set.

THEOREM 4.2. *Let G be a subset of S, then there is a smallest compact, simply-connected subset of S which contains G.*

PROOF. By Theorem 4.1, there is a smallest compact, simply-connected set A in E_2 which contains G. However, since S is a compact, simply-connected set which contains G, it is necessary that $A \subset S$. This completes the proof.

THEOREM 4.3. *Let $L_\alpha \in \mathfrak{L}_n$, then L_α is compact and simply-connected.*

PROOF. It follows from Theorem 2.3 that L_α is closed. Since $L_\alpha \subset S$ and S is compact, it follows that L_α is compact.

Now suppose that L_α is not simply-connected. Let A be the smallest simply-connected compact set containing L_α. Then there is an open set $B \neq \emptyset$ such that $B \subset A$ and $L_\alpha = A - B$. Since L_α is compact, the set A can be obtained in the following manner: Let $C(\infty)$ be the unbounded component of L_α^c, then $A = [C(\infty)]^c$. It is clear that bd $A \subset$ bd $L_\alpha \subset L_\alpha$.

To show that A is an L_n set, it is seen by Theorem 1.2 that it suffices to show that for every $p, q \in$ bd A there exists $P_n(p, q) \subset A$. Hence, let $p, q \in$ bd A. Then $p, q \in L_\alpha$ and L_α is an L_n set implies that there exists $P_n(p, q) \subset L_\alpha$. Since $L_\alpha \subset A$, it follows that $P_n(p, q) \subset A$ and hence A is an L_n set. Therefore, A is an L_n subset of S such that $L_\alpha \subsetneqq A$. This contradicts the maximality of L_α. It now follows that L_α is simply-connected.

THEOREM 4.4. *The set A_n is compact and simply-connected.*

PROOF. Since $A_n = \bigcap \mathfrak{L}_n$, the result follows easily from Theorem 4.3.

THEOREM 4.5. *Let A be a compact, L_n subset of S. Suppose $p, q \in A$ and let $C_k(p, q)$ be a minimal k-path from p to q in S. Let B be the smallest compact, simply-connected set in S containing $A \cup C_k(p, q)$, then B is an L_n set.*

PROOF. Let $x \in A$, then $p, q \in K(n, x, A)$ and thus clearly it follows that $p, q \in K(n, x, B)$. Now $C_k(p, q) \subset B$ and B is compact and simply-connected. Furthermore, it is clear that $C_k(p, q)$ is a minimal k-path from p to q in B. Therefore by Theorem 3.2, it follows that $C_k(p, q) \subset K(n, x, B)$. Notice also that bd $B \subset$ bd $A \cup C_k(p, q) \subset A \cup C_k(p, q)$. Now let $y, z \in$ bd B.

Case 1. If $y, z \in A$, then $y \in K(n, z, A)$ since A is an L_n set. It follows that $y \in K(n, x, B)$.

Case 2. If $y, z \in C_k(p, q)$, then $y \in K(n, z, C_k(p, q))$ and thus $y \in K(n, z, B)$. (Recall that $k \leqq n$.)

Case 3. Suppose that $y \in A$ and $z \in C_k(p, q)$. Since $C_k(p, q) \subset K(n, y, B)$, it follows that $z \in K(n, y, B)$.

Case 4. Suppose that $z \in A$ and $y \in C_k(p, q)$. Then as in Case 3, it is seen that $y \in K(n, z, B)$. Thus it is seen that $y, z \in$ bd B implies that $y \in K(n, z, B)$. By Theorem 1.2, it follows that B is an L_n set.

THEOREM 4.6. *If $p, q \in A_n$, then $p \in K(m, q, S)$ if and only if $p \in K(m, q, A_n)$.*

PROOF. It is clear that $p \in K(m, q, A_n)$ implies that $p \in K(m, q, S)$.

Hence, suppose that $p \in K(m, q, S)$. By Theorem 3.1, there is a minimal k-path from p to q in S for some k such that $1 \leqq k \leqq m$. Denote such a path by $C_k(p, q)$.

Let $\alpha \in \Delta_n$. Then $p, q \in A_n$ imply that $p, q \in L_\alpha$. Let B_α be the smallest compact simply-connected set in S containing $L_\alpha \cup C_k(p, q)$. By Theorem 4.5, it follows that B_α is an L_n set. Since $L_\alpha \subset B_\alpha$ and B_α is an L_n set, it follows by the maximality of L_α that $L_\alpha = B_\alpha$. In particular, it follows that $C_k(p, q) \subset L_\alpha$.

Since $\alpha \in \Delta_n$ was arbitrary, it follows that $C_k(p, q) \subset \bigcap \mathfrak{L}_n = A_n$. Thus, $p \in K(k, q, A_n)$ and since $k \leqq m$, it is seen that $p \in K(m, q, A_n)$. Thus, the desired result has been obtained.

THEOREM 4.7. *The set A_n is an L_n set.*

PROOF. Let $p, q \in A_n$ and let $L_\alpha \in \mathfrak{L}_n$. Then $p, q \in L_\alpha$ and thus it is clear that $p \in K(n, q, S)$. By Theorem 4.6, it follows that $p \in K(n, q, A_n)$. Since $p, q \in A_n$ were arbitrary, it is seen that A_n is an L_n set.

THEOREM 4.8. *The set A_n is a compact, simply-connected, L_n set.*

PROOF. Combine Theorems 4.4 and 4.7.

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