CARLEMAN AND SEMI-CARLEMAN OPERATORS

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1° **Introduction.** By a Carleman operator we mean an integral operator K on $L^2(a, b)$, $-\infty \leqq a < b \leqq +\infty$, with measurable kernel k such that

$$\int_a^b |k(x, y)|^2 dy < \infty, \quad \text{a.e. } x. \tag{1}$$

If instead of (1) we have

$$\int_a^b |k(x, y)|^2 dx < \infty, \quad \text{a.e. } y \tag{2}$$

we say K is semi-Carleman. If both (1) and (2) hold we say K is bi-Carleman. The theory of Carleman operators was initiated by T. Carleman in 1923 [1], and extended by J. von Neumann [2], using results of Weyl [3]. In the earlier literature it was always assumed that $k(x, y) = \overline{k(y, x)}$, and it is shown in Stone [4, p. 398] that with this assumption k defines (on a certain domain) a symmetric operator. In the present note we do not include the symmetry condition in our definitions, as indicated above. Carleman operators have been studied in [6], [7], [8], [9], and semi-Carleman operators in [5].

In this note we present several remarks on Carleman, semi-Carleman, and bi-Carleman operators. In 2°, we give a characterization, without reference to the kernel, of Carleman operators, with a related result concerning compactness. In 3°, we discuss the question of whether the adjoint of an operator K with kernel $k(x, y)$ is the integral operator with the transposed kernel $k^*(x, y) = \overline{k(y, x)}$. In 4° the invariance with regard to the interval employed of the representation of an operator by a semi-Carleman operator is remarked, extending previous results. In 5°, we correct an error in the paper [5].

2° A classic result of von Neumann [2], as extended in [6], states that a normal operator is unitarily equivalent to a Carleman operator if and only if 0 is a limit point in the sense of Weyl [3] of its spectrum. However, this does not characterize Carleman operators. Apart from directly exhibiting a kernel, there is the following criterion, due to

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Korotkov [11]: an operator K on $L^2(a, b)$ is a Carleman operator if and only if $|Kf(x)| \leqq g(x)$ a.e. for all f in the domain of K, $\|f\| \leqq 1$, where g is measurable and a.e. finite. Another criterion, analogous to the Hilbert-Schmidt property $(\sum \|K\phi_n\|^2 < \infty$ for $\{\phi_n\}$ complete ortho-normal) is the following.

THEOREM 2.1 (a). *A bounded operator K on $L^2(a, b)$ is Carleman if and only if*

$$\sum |K\phi_n(x)|^2 < \infty \quad \text{a.e. } x \tag{3}$$

for some complete ortho-normal system $\{\phi_n\}$.

PROOF. If K has a Carleman kernel k, then

$$k(x, \cdot) = \sum (k(x, \cdot), \bar{\phi}_n)\bar{\phi}_n, \qquad \sum |(k(x, \cdot), \bar{\phi}_n)|^2 < \infty, \quad \text{a.e. } x,$$

and $(k(x, \cdot), \bar{\phi}_n) = \int k(x, y)\phi_n(y)dy = K\phi_n(x)$, whence (3).

If (3) holds, define $l_\phi(x, y) = \sum K\phi_n(x)\overline{\phi_n(y)}$. Then l_ϕ is a Carleman kernel, and $\int l_\phi(x, y)\phi_n(y)dy = K\phi_n(x)$ for all n. The theorem now can be derived from the following.

LEMMA. *If a Carleman operator A agrees with a bounded operator B on a dense set D then $A = B$.*

PROOF. Let $a(x, y)$ be the kernel of A. For any $f \in L^2$,

$$|Af(x)| \leqq \left(\int |a(x, y)|^2 dy\right)^{1/2} \|f\| < \infty \quad \text{a.e.},$$

and if $f_n \to f$ then similarly $Af_n(x) \to Af(x)$ a.e. Now let $f \in L^2$ be arbitrary, $f_n \to f$, $f_n \in D$. Then $Af_n \to Af$ a.e., and $Bf_n \to Bf$ in norm. But then a subsequence $Bf_{n_k} \to Bf$ a.e., whence $Af = Bf$. Q.E.D.

Returning to the proof of Theorem 2.1 (a), we have the Carleman operator with kernel l_ϕ agreeing with the bounded operator K on the span of the system $\{\phi_n\}$. Hence by the lemma, the two are equal, and the proof is complete.

REMARK. If (3) holds for one system $\{\phi_n\}$ it holds for all. For the unbounded case we have

THEOREM 2.1 (b). *Let the operator K on $L^2(a, b)$ have dense domain D_K. K coincides with a Carleman operator L on a dense set $D \subset D_K \cap D_L$ if and only if* (3) *holds for some complete orthonormal set* $\{\phi_n\} \subset D_K$.

PROOF. Since D_K is dense, we can choose a countable dense set in D_K, and apply the Gram-Schmidt process to arrive at an orthonormal set in D_K. By density it will be complete.

If K coincides with a Carleman operator L on a dense set $D \subset D_K \cap D_L$, choose a system $\{\phi_n\} \subset D$ and find (as in Theorem 2.1 (a)) $\sum |K\phi_n(x)|^2 < \infty$ a.e. x.

If (3) holds, define l_ϕ as before, and observe that the integral operator with kernel l_ϕ coincides with K on the linear span of the system $\{\phi_n\}$.

A related result is

THEOREM 2.2. *Suppose A is a normal Carleman operator on $L^2(a, b)$, $-\infty < a < b < \infty$, having a complete ortho-normal set of eigenvectors $\{\phi_n\}$. If there exists a function $\psi \in L^2$ such that either*

(i) $|\phi_n(x)| \leqq |\psi(x)|$ *for all n and* a.e. x, *or*

(ii) $|A\phi_n(x)| \leqq |\psi(x)|$ *for all n and* a.e. x,

then A is compact.

PROOF. We use the fact that a normal operator whose spectrum has no limit point but 0 is compact. Let $A\phi_n = \lambda_n\phi_n$. By Theorem 2.1, $\lim |A\phi_n(x)| = \lim |\lambda_n||\phi_n(x)| = 0$, a.e. x.

Case (i). If A is not compact, there exists a subsequence λ_{n_k} with $|\lambda_{n_k}| \geqq \delta > 0$. Hence $|\phi_{n_k}(x)| \to 0$, so by dominated convergence $\|\phi_{n_k}\| \to 0$, but $\|\phi_{n_k}\| = 1$.

Case (ii). Here $\|A\phi_n\|^2 = |\lambda_n|^2\|\phi_n\|^2 \to 0$ by dominated convergence, and $\|\phi_n\| = 1$, so $\lambda_n \to 0$.

3° When is the adjoint K^* of an integral operator K with kernel k the integral operator with the kernel $k^*(x, y) = \overline{k(y, x)}$? Two sufficient conditions are

$$\iint dxdy\, |k(x, y)|^2 < \infty, \tag{4}$$

and

$$\int |k(x, y)||f(y)|\, dy \in L^2(a, b) \quad \text{for all } f \in L^2(a, b). \tag{5}$$

The second is Zaanen's property P [**10**, p. 227]. In both cases the result follows by applying Fubini's theorem to the double integral (Kf, g). Several weaker statements are possible.

THEOREM 3.1. *If K is semi-Carleman with finite rank kernel k then K^* has kernel k^*.*

PROOF. By assumption

$$k(x, y) = \sum_1^n \phi_j(x)\overline{\psi_j(y)},$$

where the ϕ_j are ortho-normal and the ψ_j need not be L^2 but $\sum |\psi_j(y)|^2 < \infty$, a.e. y. For f in the domain D_K of K (necessarily dense; see [5]) we have $(Kf, g) = (\sum (f, \psi_j)\phi_j, g)$. Since $f \in D_K$ all $|(f, \psi_j)| < \infty$. Therefore $(f, \sum(\phi_j, g)\psi_j)$ is finite and equal to $\sum(g, \psi_j)(\phi_j, g) = (Kf, g)$. Since $D_{K^*} = \{g \mid \exists h \forall f \in D_K \quad (Kf, g) = (f, h)\}$, we have $h = K^*g = \sum(\phi_j, g)\psi_j$, so that $g \in D_{K^*}$ if and only if $\sum(\phi_j, g)\psi_j \in L^2$. Thus $D_{K^*} \neq \{0\}$, for it contains in particular all g which are orthogonal to the ϕ_j. For $g \in D_{K^*}$ we have $K^*g(x) = \int k^*(x, y)g(y)dy$, as was to be shown.

THEOREM 3.2. *If the adjoint of a bounded Carleman operator is Carleman then the kernel of the adjoint is the transposed conjugate kernel, and thus the operator is bi-Carleman.*

PROOF. Let K be a bounded Carleman operator with kernel k, and suppose K^* has a Carleman kernel l. Then $\mathrm{Re}(K) = \frac{1}{2}(K + K^*)$ has kernel $\frac{1}{2}\{k(x, y) + l(x, y)\}$ and is selfadjoint. By a recent result of J. Weidmann [12], the kernel of a bounded self adjoint Carleman operator is symmetric. Hence $(1/2)\{k+l\}$ and $(1/2i)\{k+l\}$ are both symmetric, whence $k^* + l^* = k + l$ and $-k^* + l^* = k - l$. Adding, we have $k^* = l$. Q.E.D.

It follows from this that a bounded normal Carleman operator is bi-Carleman, since it is known [6] that the adjoint of a normal Carleman operator is Carleman.

THEOREM 3.3. *The adjoint of a bounded bi-Carleman operator is represented by the transposed kernel, and is therefore again a bi-Carleman operator.*

PROOF. Let K be bounded and bi-Carleman, with kernel k. Let $k^t(x, y) = \overline{k(y, x)}$ and K^t the operator with kernel k^t. Now K^t is semi-Carleman in particular, and so $\{f \mid K^t f \in L^2\}$ is dense (see [5]). Then $\frac{1}{2}(K + K^t)$ is densely defined and has symmetric Carleman kernel $\frac{1}{2}(k + k^t)$. It follows that there exists a dense set $D \subset L^2$ on which $K + K^t$ is defined and symmetric [4, p. 398], and on which $K - K^t$ is antisymmetric. Now $K = \frac{1}{2}(K + K^t) + \frac{1}{2}(K - K^t)$ on D, and $K^* = \frac{1}{2}(K + K^t) - \frac{1}{2}(K - K^t)$ on D, so that $K + K^* = K + K^t$ on D. Hence $K^* = K^t$ on D. Thus we have a bounded operator (K^*) equal to a Carleman operator (K^t) on the dense domain D. By the lemma of 2°, $K^* = K^t$ on all of L^2, and the proof is complete.

These results have a stronger form, with different proofs, in a forthcoming paper of Weidmann [12].

An example of a bi-Carleman operator which is not of Hilbert-Schmidt class is afforded by the operator of convolution on $L^2(-\infty, \infty)$ by an L^2 function.

That the adjoint of this operator is the integral operator with the transposed conjugate kernel may be seen by using the Fourier transform F as follows. Let L_ϕ denote convolution by $\phi \in L^2$, and let M_k denote multiplication by the function k. We have $FL_\phi F^{-1} = M_{F\phi}$ (from the relation $F(f_1 * f_2) = (Ff_1)(Ff_2)$). Taking adjoints we have $F(L_\phi)^* F^{-1} = M_{\overline{F\phi}}$. Let $\tilde{\phi}(x) = \overline{\phi(-x)}$. Then $FL_{\tilde{\phi}}F^{-1} = M_{F\tilde{\phi}}$. If we show $M_{F\tilde{\phi}} = M_{\overline{F\phi}}$ then we will have shown that $(L_\phi)^* = L_{\tilde{\phi}}$. To show $M_{F\tilde{\phi}} = M_{\overline{F\phi}}$ it suffices to show $\overline{F\phi} = F\tilde{\phi}$. But this is easily seen by direct verification. Now

$$(L^{\tilde{}}f)(x) = \int \overline{\phi(y - x)} f(y) dy,$$

as was to be shown.

4° **Invariance.** In [6] it was shown that certain unitary transformations introduced by von Neumann in [2] preserve the Carleman property (1) of an integral operator. In this section we observe that the same is true for the semi-Carleman property (2). The proof is straightforward, and follows the steps of [6, Appendix 2]. We shall here sketch the main points.

The transformation

$$(U_{a,b}f)(x) = (b - a)^{1/2} f\left(\frac{x - a}{b - a}\right) \tag{6}$$

maps $L^2(0, 1)$ onto $L^2(a, b)$ unitarily, and clearly preserves (2). That is, a kernel $k(x, y)$ on $L^2(0, 1)$ becomes essentially

$$k\left(\frac{u - a}{b - a}, \frac{v - a}{b - a}\right)$$

on $L^2(a, b)$, and so integrability properties are preserved.

The transformation

$$(N_1 f)(x) = \frac{1}{\sqrt{x}} f(\log x) \tag{7}$$

maps $L^2(-\infty, \infty)$ onto $L^2(0, \infty)$ unitarily, and

$$(N_2 f)(x) = \frac{1}{1 - x} f\left(\frac{x}{1 - x}\right) \tag{8}$$

maps $L^2(0, \infty)$ onto $L^2(0, 1)$ unitarily. Now a semi-Carleman kernel $k(x, y)$ on $L^2(-\infty, \infty)$ becomes $1/(uv)^{1/2} k(\log u, \log v)$ on $L^2(0, \infty)$, and

$$\int_0^\infty \frac{1}{uv} \mid k(\log u, \log v) \mid^2 du = \frac{1}{v} \int_{-\infty}^\infty \mid k(w, \log v) \mid^2 dw < \infty$$

for a.e. $v \in [0, \infty]$, whence the new kernel is semi-Carleman on $L^2(0, \infty)$. Similarly, $k(x, y)$ given and semi-Carleman on $L^2(0, \infty)$ becomes

$$k\left(\frac{u}{1-u}, \frac{v}{1-v}\right) \cdot \frac{1}{(1-u)(1-v)}$$

on $L^2(0,1)$, and

$$\int_0^1 \left| k\left(\frac{u}{1-u}, \frac{v}{1-v}\right) \frac{1}{(1-u)(1-v)} \right|^2 du$$
$$= \frac{1}{(1-v)^2} \int_0^\infty \left| k\left(w, \frac{1}{1-v}\right) \right|^2 dw < \infty$$

for a.e. $v \in [0, 1]$.[2] Composing these transformations, and using the known result for Carleman operators, we have the following statement.

THEOREM 4.1. *An operator is unitarily equivalent to a Carleman (semi-Carleman), (bi-Carleman) operator on $L^2(-\infty, \infty)$ if and only if it is unitarily equivalent to a Carleman (semi-Carleman) (bi-Carleman) operator on $L^2(a, b)$, where (a, b) is any finite or semifinite interval.*

5° **Semi-Carleman representation.** In [5] it was falsely claimed that the adjoint of a semi-Carleman operator is densely defined. The error was pointed out by J. Weidmann (private communication). His remark is the following. Let $k(x, y) = \alpha(x)\overline{\beta(y)}$, with $\alpha \in L^2$ and β locally square integrable but not square integrable, and let K be the operator with kernel k. By the proof of Theorem 3.1 specialized to the rank 1 case, we have $D_{K^*} = \{g \mid (g, \alpha) = 0\}$, which is not a dense set. Because of this, it remains an open question whether semi-Carleman operators have closed extensions. Therefore the final statement of [5, §4], should read: A closed densely defined operator is representable on $L^2(-\infty, \infty)$ as a semi-Carleman operator if and only if it has 0 as a limit point of its spectrum.

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[2] Corresponding statements for unitary mappings from $[-\infty, a]$ and $[a, \infty]$ to $[0, \infty]$ are easily proved.

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