A CLASS OF COUNTABLY PARACOMPACT SPACES

PHILLIP ZENOR

A space X is said to have property $\mathcal{B}$ if for any well-ordered monotone decreasing family $\{H_a | a \in A\}$ of closed sets with no common part, there is a monotone decreasing family of domains $\{D_a | a \in A\}$ such that

(i) $H_a \subset D_a$ for each a in A and

(ii) $\{\mathrm{cl}(D_a) | a \in A\}$ has no common part.

It is shown that property $\mathcal{B}$ characterizes the separable T_3-spaces that are Lindelöf and the countably compact spaces that are compact. Also, it is shown that the T_3-space X is Lindelöf if and only if X has property $\mathcal{B}$ and every uncountable subset of X has a limit point.

Throughout this paper, topological spaces are assumed to be T_1-spaces.

1. Preliminary results and lemmas.

1.1. If X has property $\mathcal{B}$, then X is countably paracompact.

This is immediate from [1] where it is shown that X is countably paracompact if and only if for any countable decreasing sequence of closed sets $\{H_n\}$ with no common part, there is a monotone decreasing sequence of domains $\{D_n\}$ such that

(i) for each n, $H_n \subset D_n$ and

(ii) $\{\mathrm{cl}(D_n)\}$ has no common part.

1.2. If X is paracompact, then X has property $\mathcal{B}$.

PROOF. Let $\{H_a | a \in A\}$ denote a well-ordered, monotone family of closed sets with no common part. Then $\{G_a = X - H_a | a \in A\}$ is an open cover of X. Hence, there is a locally finite open refinement $\{G'_a | a \in A\}$ of $\{G_a | a \in A\}$ such that $G'_a \subset G_a$ for each a in A. For each a in A, let $D_a = \bigcup\{G'_b | b \in A,\ b \geqq a\}$. $\{D_a | a \in A\}$ satisfies the conditions for property $\mathcal{B}$.

1.3. THEOREM. *If the T_2-space X has property $\mathcal{B}$, then X is T_3.*

PROOF. Suppose the contrary; that is, suppose that there is a closed set H and a point P not in H such that if O is an open set containing H, then P is in $\mathrm{cl}(O)$. Let G be an open cover of H of minimal cardinal ρ such that if g is in G then $\mathrm{cl}(g)$ does not contain P. Note that it follows from the supposition that ρ cannot be finite.

Received by the editors October 11, 1968.

[1] Part of these results were presented to the American Mathematical Society, January 25, 1969. This research was partially supported by NASA Grants NsG(T)-52 and NGR-44-005-037. Part of these results are contained in the author's doctoral thesis at the University of Houston.

Let $\{g_a | a \in A\}$ be a well-ordering of G according to the initial ordinal of cardinal ρ. Then $\{h_a = H - \bigcup_{b<a} g_b | a \in A\}$ is a well-ordered monotone decreasing family of closed sets with no common part. Since S has property $\mathfrak{B}$, there is a domain D containing $h_{a'}$, for some a' in A, such that P does not belong to $\mathrm{cl}(D)$. Hence, $G' = \{g_b | b \leqq a'\} \cup D$ is an open cover of H such that if $g \in G'$, then P is not in $\mathrm{cl}(g)$. But the cardinality of G' is less than ρ which is a contradiction from which the theorem follows.

The following example, brought to the attention of the author by John Mack, shows that property $\mathfrak{B}$ cannot be replaced by countable paracompactness in Theorem 1.3.

Let $[0, \Omega)$ denote the segment of countable ordinals, where Ω denotes the first uncountable ordinal and let $[0, \Omega] = [0, \Omega) \cup \{\Omega\}$. Let $Y = [0, \Omega] \times [0, \Omega] - (\Omega, \Omega)$. Then $Y/(\Omega \times [0, \Omega))$ is a countably compact (and therefore countably paracompact) T_2-space that is not T_3.

Related to Theorem 1.3 are the following questions:

1. If X is a T_2-space with property $\mathfrak{B}$ such that each closed set is a G_δ-set, then is X normal?

2. If X is a T_2-space with property $\mathfrak{B}$ such that each closed subset of X is a G_δ-set, then is X hereditarily countably paracompact?

With techniques similar to those used in [2], Questions 1 and 2 can be shown to be equivalent.

Recall that the function f from X to Y is said to be a *proper* mapping if f is continuous, closed, and $f^{-1}(y)$ is compact for each y in Y.

1.4. THEOREM. *If X has property $\mathfrak{B}$ and f is a proper mapping from X onto Y, then Y has property $\mathfrak{B}$.*

PROOF. Let $\{H_a | a \in A\}$ be a well-ordered, monotone decreasing family of closed sets in Y with no common part. Then $\{f^{-1}(H_a) | a \in A\}$ is a well-ordered monotone decreasing family of closed sets in X with no common part. Let $\{D_a | a \in A\}$ denote the family of domains in X given by property $\mathfrak{B}$ for $\{f^{-1}(H_a) | a \in A\}$. Let $O_a = Y - f(X - D_a)$ for each a in A. Then $\{O_a | a \in A\}$ is a well-ordered family of domains in Y such that $H_a \subset O_a$ for each a in A. Suppose that $y \in \bigcap_{a \in A} \mathrm{cl}(O_a)$. Then $f^{-1}(y)$ is a compact set that intersects $\mathrm{cl}(D_a)$ for each a in A which is impossible; and so, $\{\mathrm{cl}(O_a) | a \in A\}$ has no common part.

1.5. COROLLARY. *If X has property $\mathfrak{B}$ and Y is compact, then $X \times Y$ has property $\mathfrak{B}$.*

1.6. LEMMA. *If X has property $\mathfrak{B}$ and $\{K_a | a \in A\}$ is a well-ordered, countably centered, monotone decreasing family of closed sets in X with no common part, then there is an uncountable pair-wise disjoint family of nonempty domains and an uncountable, closed discrete subset of X.*

PROOF. For each a in A, let

$$F_a = \bigcap_{b<a} K_b \qquad \text{if } a \text{ is a limit ordinal,}$$
$$= K_a \qquad \text{otherwise.}$$

Then $\{F_a | a \in A\}$ is a well-ordered, countably centered, monotone decreasing family of closed sets with no common part. Let $\{D_a | a \in A\}$ be the collection of domains given by property $\mathcal{B}$ for $\{F_a | a \in A\}$. For each a in A, the set $\{b \in A | D_a - \mathrm{cl}(D_b) \neq \varnothing\}$ is not empty. Let τ be the function from A into A that takes a into the first element of $\{b \in A | D_a - \mathrm{cl}(D_b) \neq \varnothing\}$. Observe that for each a in A, $\tau(a) > a$.

Let θ denote the function taking A into the power set of X by letting $\theta(0) = D_0 - \mathrm{cl}(D_{\tau(0)})$ and

$$\theta(b) = D_b - \mathrm{cl}(D_{\tau(b)}) \qquad \text{if} \quad \sup\{\tau(a) \mid a < b\} \leqq b,$$
$$= \varnothing \qquad \text{otherwise.}$$

Clearly $\theta(A)$ is a pair-wise disjoint collection of domains. If it can be shown that $A' = \{a \in A | \theta(a) \neq \varnothing\}$ is cofinal in A, it will follow that $\theta(A)$ is uncountable; otherwise, $\{F_a | a \in A'\}$ would be a countable subcollection of $\{F_a | a \in A\}$ with no common part. To see that A' is cofinal in A, suppose the contrary; that is, suppose that $b = \sup A'$ is in A. Let $b_1 = \tau(b)$ and, proceeding by induction, let $b_{n+1} = \tau(b_n)$. The set $\{b_n | n = 1, 2, \cdots\}$ is not cofinal in A, since $\{F_b\}$ is countably centered, so $b_0 = \sup\{b_n\}$ is in A. Since τ is monotone nondecreasing, it follows that $\sup\{\tau(a) | a < b_0\} \leqq b_0$; and so b_0 is in A'.

For each a in A', let P_a be a point of $\theta(a)$. To see that $\{P_a | a \in A'\}$ has no limit point, suppose the contrary; that is, suppose that P is a limit point of $\{P_a | a \in A'\}$. Since if a is a limit ordinal of A, $F_a = \bigcap_{b<a} F_b$, there is a last element a' of A such that $F_{a'}$ contains P. Let $\mathcal{P}_1 = \{P_a | a \in A', a \leqq a'\}$ and $\mathcal{P}_2 = \{P_a | a \in A', a > a'\}$.

Since P does not belong to $F_{a'+1}$, P is not a limit point of $\mathcal{P}_2$; and so, P must be a limit point of $\mathcal{P}_1$. If $\{a \in A' | a \leqq a'\}$ has no last term, $D_{a'}$ is a domain containing P but no points of $\mathcal{P}_1$; hence, $\{a \in A' | a \leqq a'\}$ must have a last term, say b'. Then $D_{b'}$ is a domain containing P that contains only one point of $\mathcal{P}_1$, namely $P_{b'}$, which is a contradiction from which it follows that $\{P_a | a \in A'\}$ has no limit point.

1.7. LEMMA. *Suppose that X is a space with an uncountable, closed, discrete subspace H. If X has property $\mathcal{B}$, then there are uncountably many mutually exclusive nonempty domains in X.*

PROOF. Let K denote a subcollection of H with cardinality $\aleph_1$. Let $\{P_a | a \in A\}$ be a well-ordering of K according to the least ordinal of cardinal $\aleph_1$. For each a in A, let $K_a=\{P_b \mid b\in A,\ b\geqq a\}$. Then $\{K_a \mid a\in A\}$ is a well-ordered, monotone decreasing family of closed sets with no common part. Since the cardinality of K is $\aleph_1$, $\{K_a \mid a\in A\}$ is countably centered; therefore, by Lemma 1.6, there is an uncountable family of mutually exclusive, nonempty domains in X.

A space X is said to have the *Souslin property* if there is no uncountable collection of mutually exclusive nonempty domains in X.

1.8. COROLLARY. *If the space S has the Souslin property and property $\mathcal{B}$, then every uncountable subset of X has a limit point.*

1.9. COROLLARY. *If X is a separable space with property $\mathcal{B}$, then every uncountable subset of X has a limit point.*

The following result, due to W. B. Sconyers [**3**, Theorem 3], is stated as a lemma.

1.10. LEMMA. *The T_3-space X is Lindelöf if and only if for each well-ordered, monotone increasing family $\mathfrak{D}$ of domains covering the space, there is a countable collection of closed sets that refines $\mathfrak{D}$ and covers X.*

2. Main results.

2.1. THEOREM. *The T_3-space X is Lindelöf if and only if*
(i) *X has property $\mathcal{B}$ and*
(ii) *every uncountable subset of X has a limit point.*

PROOF. It is well known that if X is Lindelöf, then X is paracompact; and so, by 1.2, X has property $\mathcal{B}$.

Suppose that (i) and (ii) are satisfied. By Lemma 1.10, it is sufficient to show that there is a countable collection, $\{F_n\}$, of closed sets refining $\{D_a \mid a\in A\}$ and covering X, where $\{D_a \mid a\in A\}$ is a well-ordered, monotone increasing open cover of X. It follows easily from Lemma 1.6 that there is a countable subset B of A such that $\{D_b \mid b\in B\}$ covers X. Hence, $\{E_b=X-D_b \mid b\in B\}$ is a countable, well-ordered family of closed sets with no common part. Let $\{G_b \mid b\in B\}$ be the domains given for $\{E_b \mid b\in B\}$ by property $\mathcal{B}$. Then $\{F_b=X-G_b \mid b\in B\}$ is the desired collection of closed sets.

2.2. COROLLARY. *The countably compact T_3-space X is compact if and only if X has property $\mathcal{B}$.*

The following theorems follow immediately from Theorem 2.1, Lemma 1.7 and Corollary 1.8:

2.3. THEOREM. *If the T_3-space X has the Souslin Property, then X is Lindelöf if and only if X has property $\mathcal{B}$.*

2.4. THEOREM. *The separable T_3-space X is Lindelöf if and only if X has property $\mathcal{B}$.*

REFERENCES

1. F. Ishikawa, *On countably paracompact spaces*, Proc. Japan Acad. **31** (1955), 686–687. MR **17**, 650.

2. P. Zenor, *On countable paracompactness and normality*, Prace Mat. (to appear).

3. W. B. Sconyers, *Concerning the Lindelöf property*, Notices Amer. Math. Soc. **15** (1968), 342. Abstract #654-19.

UNIVERSITY OF HOUSTON AND
AUBURN UNIVERSITY