A RESULT OF BASS ON CYCLOTOMIC EXTENSION FIELDS[1]

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In [1] Bass stated the result given below as Proposition 1 and derived some consequences. His proof of the proposition itself, however, contains a gap; Lemmas 2 and 3 are false as stated. The purpose of this note is to fill the gap by proving the slightly stronger Proposition 2.

We retain the notation of [1]. In particular $k_m = k(\zeta_m)$ where $\zeta_m = e^{2\pi i/m}$. The letters $m, n, a, b, c, d, r, s, t, u, v$ will denote nonnegative integers, p is a prime integer, and $K = k(i)$.

PROPOSITION 1. *Given k and n, there is an m such that $k_m^{*^m} \cap k^* \subset k^{*^n}$.*

PROPOSITION 2. *Given k there is an m such that for all n, $k_{mn}^{*^{mn}} \cap k^* \subset k^{*^n}$.*

LEMMA 1. *Suppose $i \in k$ if $p = 2$. Then if $r = p^a$, $k_r^{*^r} \cap k^* \subset k^{*^r}$.* (For proof see p. 39 of [2].)

LEMMA 2. *Given p and k with $i \in k$ if $p = 2$, suppose $r = p^a$ and v are such that $\zeta_{pr} \notin k_v$. Then for all $t = p^c$, $k_v^{*^{rt}} \cap k^* \subset k^{*^t}$.*

PROOF. If $c = 0$ the result is trivial; assume $c > 0$.

Case 1. $\zeta_p \in k$ or $\zeta_p \notin k_v$.

For any $u = p^d$, $d > 0$, any ruth power, $z \in k^*$ of an element in k_v^* is a pth power of an element in k^*. If not, $X^{ru} - z$ would be irreducible over k [3, p. 221], hence all its roots would lie in k_v, which is normal over k, hence $\zeta_{ru} \in k_v$, contrary to supposition.

Therefore, if $x = y^{rt}$, $x \in k$, $y \in k_v^*$, then $x = w^p$, $w \in k^*$, and $w^{-1} y^{rt/p}$ is a pth root of 1 in k_v, hence in k, and $y^{rt/p} \in k$. Repeating the argument if necessary we conclude, $y^r \in k^*$, $x = y^{rt} \in k^{*^t}$.

Case 2. $\zeta_p \notin k$, $\zeta_p \in k_v$.

If $x \in k$ is an rtth power of something in k_v then by Case 1, x is a tth power of something in $k_p \subset k_v$. Taking norms from k_p to k and noting that $[k_p : k]$ is prime to t gives the result.

LEMMA 3. *Let $s = 2^b$ be such that $\zeta_{2s} \notin K$. Then for any $t = 2^c$, $K^{*^{st}} \cap k^* \subset k^{*^t}$.*

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PROOF (FOLLOWING [2]). Let $x = y^{st}$, $x \in k^*$, $y \in K^*$. If $y \in k^*$ there is nothing to prove, so assume $u = 2^d$ such that $y^u \notin k^*$, $y^{2u} \in k^*$. Then $y^u = iz$, $z \in k^*$ and if σ denotes conjugation over k, $(y^{-1}y^\sigma)^u = -1$. Hence $u < s$, $y^s \in k^*$, $x = y^{st} \in k^{*^t}$.

We are now ready to prove Proposition 2. For all ramified odd p let a_p denote one plus the exponent of p in the ramification degree, from Q to k, of some prime dividing p; for unramified odd p let $a_p = 0$, and let a_2 be one plus the exponent of 2 in the ramification degree, from Q to K, of some prime dividing 2. Let $r_p = p^{a_p}$. Then for all p and v prime to p, $\zeta_{pr_p} \notin k_v$, in fact $\zeta_{2r_2} \notin K_v$. Let $s_p = r_p$ for p odd and $s_2 = r_2^2$, and let $m = \Pi s_p$. Then for any $n = \Pi t_p$, $t_p = p^{c_p}$, letting $u_p = s_p t_p$,

$$
\begin{aligned}
k_{mn}^{*^{mn}} \cap k^* &= \Big(\bigcap_p k_{mn}^{*^{u_p}}\Big) \cap k^* \\
&\subset \Big(\bigcap_{p \neq 2} k_{mn}^{*^{u_p}} \cap k^*_{mn/u_p}\Big) \cap \big(K_{mn}^{*^{u_2}} \cap K_{mn/u_2}\big) \cap k^* \\
&\subset \Big(\bigcap_{p \neq 2} k_{mn/u_p}^{*^{u_p}}\Big) \cap \big(K_{mn/u_2}^{*^{u_2}}\big) \cap k^* && \text{(by Lemma 1)} \\
&\subset \Big(\bigcap_{p \neq 2} k^{*^{t_p}}\Big) \cap \big(K^{*^{r_2 t_2}}\big) \cap k^* && \text{(by Lemma 2)} \\
&\subset \bigcap_p \big(k^{*^{t_p}}\big) = k^{*^n} && \text{(by Lemma 3).}
\end{aligned}
$$

PROPOSITION 3. *If* $E = 2D_{k/Q}$, then $\bigcap_r k_{E^r}^{*^{E^r}} = \{1\}$.

PROOF. k contains no nontrivial roots of unity of order prime to E. Hence if $x \in k^*$, $x \neq 1$, $x \notin k^{*^s}$ for some $s = E^b$. The only odd primes in the m of Proposition 2 are ramified ones, hence $m \mid t$ for some $t = E^c$. Then $x \notin k_{st}^{*^{st}} \cap k^* \subset k^{*^{st/m}} \subset k^{*^s}$.

REFERENCES

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