LARGE DEVIATION PROBABILITIES FOR POSITIVE RANDOM VARIABLES

G. R. ANDERSEN

1. **Introduction.** Let $X_1, X_2, \cdots$ be a sequence of positive, independent and identically distributed random variables (r.v.'s) with distribution function (d.f.) F. Set $S_n = \sum_{K=1}^n X_K$. Following Heyde [2], we call $P(S_n > t_n)$ a large deviation probability when $t_n \to +\infty$ as $n \to +\infty$.

2. Results. The following theorem sharpens Heyde's result at the expense of a restriction to positive r.v.'s with slowly varying tail distributions.

THEOREM. *Let $\{t_n\}$ be a sequence of positive numbers with $t_n \to +\infty$ in such a way that $n(1-F(t_n)) \to 0$ as $n \to +\infty$. If $1-F$ is a slowly varying function, then*

$$P(S_n > t_n) \sim nP(X_1 > t_n) \sim P(\max_{i \leq n} X_i > t_n), \tag{1}$$

as $n \to +\infty$.

The right-hand side of (1) is a consequence of the following

LEMMA. *If $n(1-F(t_n)) \to 0$ as $n \to +\infty$, then*

$$nP(X_1 > t_n) \sim P(\max_{i \leq n} X_i > t_n), \qquad (n \to +\infty). \tag{2}$$

Heyde, [2], obtains this result from Bonferroni's inequalities; it is also a simple consequence of the following inequality:

$$n(1 - F(t_n)) \geqq 1 - F^n(t_n) \geqq 1 - e^{-n(1-F(t_n))}.$$

PROOF OF THE THEOREM. Let $M_n = \max_{i \leq n} X_i$. It is easy to see that

$$P(S_n > t_n) = P(M_n > t_n) + P(M_n \leqq t_n, S_n > t_n). \tag{3}$$

Hence, using only the hypothesis of the lemma and (2), we find that

$$\liminf_n \{P(S_n > t_n)/nP(X_1 > t_n)\} \geqq 1. \tag{4}$$

In order to prove the other half of (1) we let S_{nn} be the sum of the truncates of $X_1, X_2, \cdots, X_n$ at the point t_n. Then, from equation (3),

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$$P(S_n > t_n) \leqq P(M_n > t_n) + P(S_{nn} > t_n). \tag{5}$$

Applying Markov's inequality to $P(S_{nn} > t_n)$ we find that

$$P(S_{nn} > t_n) \Big/ \{nP(X_1 > t_n)\} \leqq \int_0^{t_n} x dF(x) \Big/ \{t_n P(X_1 > t_n)\}. \tag{6}$$

Since $1-F$ is a slowly varying function, we may use Karamata's theorem [**1**, p. 273], to obtain the following limit.

$$\int_0^{t_n} (1 - F(y)) dy / \{t_n P(X_1 > t_n)\} \to 1,$$

as $n \to +\infty$. Then, after integrating by parts to obtain

$$\int_0^{t_n} y dF(y) = -t_n P(X_1 > t_n) + \int_0^{t_n} (1 - F(y)) dy,$$

we see that the right-hand side of (6) approaches zero as $n \to +\infty$. Hence, $P(S_{nn} > t_n) = o(nP(X_1 > t_n))$, $(n \to +\infty)$. But then the lemma and inequality (5) imply that

$$\limsup_n (P(S_n > t_n)/\{nP(X_1 > t_n)\}) \leqq 1. \tag{7}$$

REMARK 1. The proof of the theorem goes through if $t_n \to +\infty$ independently of n. That is, *if* $1-F$ *is slowly varying then* $P(S_n > t) \sim nP(X_1 > t) \sim P(\max_{i \leqq n} X_i > t)$ *as* $t \to +\infty$ *for each* n. This known result is given in [**1**, p. 272].

REMARK 2. The converse of the theorem is not true; it is easy to show that (1) holds for the one-sided stable law with parameter $1/2$: $F(x) = 2(1 - \Phi(1/\sqrt{x}))$, $x > 0$, where Φ is the standard normal d.f., provided that $n^2 = o(t_n)$ $(n \to +\infty)$. However, $1-F$ is a regularly varying function with exponent $-1/2$.

REMARK 3. If $1-F$ is regularly varying with exponent $-\gamma$, $0 < \gamma < 1$, then the above argument yields

$$\begin{aligned} 1 &\leqq \liminf_n \{P(S_n > t_n)/nP(X_1 > t_n)\} \\ &\leqq \limsup_n \{P(S_n > t_n)/nP(X_1 > t_n)\} \\ &\leqq (1 - \gamma)^{-1}, \end{aligned}$$

if $nP(X_1 > t_n) \to 0$, as $n \to +\infty$.

REFERENCES

1. W. Feller, *An introduction to probability theory and its applications*. Vol. II, Wiley, New York, 1966. MR **35** #1048.

2. C. C. Heyde, *On large deviation problems for sums of random variables which are not attracted to the normal law*, Ann. Math. Statist. **38** (1967), 1575–1578. MR **36** #4616.

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