A NOTE ON MOUFANG VEBLEN-WEDDERBURN SYSTEMS

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ABSTRACT. The purpose of this note is to show that a Veblen-Wedderburn system with multiplicative Moufang identity is a near field if its dimension d over its kern does not exceed 7.

1. **Introduction.** In a recent paper [2] Kallaher investigates (left) Veblen-Wedderburn systems (and the corresponding projective planes) in which the Moufang identity

$$(x \cdot y) \cdot (z \cdot x) = (x \cdot (y \cdot z)) \cdot x \tag{1}$$

holds. Such a system is called a Moufang (left) Veblen-Wedderburn system (MVW system). Fields, near fields and Cayley-Dickson algebras are examples of MVW systems. A proper MVW system is one in which the other distributive law does not hold. The only proper MVW systems known are the near fields. Kallaher [2] obtains two sets of conditions under which an MVW system is a near field. Recently the author has been able to show [3] that there are no proper finite MVW systems other than the near fields. The object of this note is to extend this result to the infinite systems of dimension $d \leqq 7$ over their kerns.

2. During the course of the proof of Theorem 1 the following results are needed. Proofs of these results may be found in the references indicated.

RESULT 1. Every two elements of a Moufang loop generates a subgroup (di-associativity) (Bruck [1]).

RESULT 2. Let $F(+, \cdot)$ be an MVW system with kern K. If $A(\cdot)$ is a maximal associative subloop contained in the loop $F'(\cdot)$, then $B(+, \cdot)$ is a maximal near field contained in $F(+, \cdot)$ where $B = A \cup \{0\}$ and F' consists of all nonzero elements from F. Further B contains K [3, Lemmas 3.1 and 3.2].

THEOREM 1. *Let $F(+, \cdot)$ be a left* MVW *system of dimension d over its kern K (as a right vector space). If $d \leqq 7$, then $F(+, \cdot)$ is a near field.*

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Proof. In the course of this proof we use freely the right inverse property, left distributive law and some properties of the kern K. Also we write ab in place of $a\cdot b$. If $d=1$, then $F=K$ and the theorem is obvious. Suppose $1<d\leqq 7$. Let x be an element from F which does not belong to K and $G=\langle x\rangle$ be the subloop of $F'(\cdot)$ generated by x. G is obviously an associative subloop of $F'(\cdot)$ and consequently there exists a maximal associative subloop A in $F'(\cdot)$ which contains G. From Result 2 it follows that $B(+,\ \cdot)$ is a maximal near field where $B=A\cup\{0\}$. If $B=F$, the theorem is proved. Suppose $B<F$. Then there is an element y in F such that $y\notin B$. Let $H=\langle x, y\rangle$, the subloop of $F'(\cdot)$ generated by x and y (H exists since $F'(\cdot)$ is di-associative). Let M be a maximal associative subloop of $F'(\cdot)$ containing H. Using Result 2 again we obtain that $N(+,\ \cdot)$ is a near field where $N=M\cup\{0\}$. We claim that $N=F$ and consequently $F(+,\ \cdot)$ is a near field. Suppose $N<F$. Then there is an element z in F which does not belong to N. We now show that the existence of z leads to the conclusion that the set $T=\{1, x, y, yx, z, zx, zy, z(yx)\}$ is independent over K implying a contradiction that $F(+,\ \cdot)$ is of dimension $d\geqq 8$ over its kern. Firstly we show that the set $\{1, x, y, yx\}$ is independent over K. Suppose there are elements a, b, c and k in K such that $a+xb+yc+yxk=0$. Then it follows that $y(c+xk)=-(a+xb)$. Suppose $c+xk\neq 0$. Then $y=(-(a+xb))(c+xk)^{-1}\in B$ a contradiction to the choice of y. Thus $a+xb=0$ and consequently $c+xk=0$ which imply that $a=b=c=d=0$. Hence the set $\{1, x, y, yx\}$ is independent over K.

Suppose there are elements a, b, c, k, e, f, g and h in K such that $a+xb+yc+yxk+ze+zxf+zyg+z(yx)h=0$. This relation may be rewritten as $zX=Y$ where $X=(e+xf+yg+yxh)$ and $Y=-(a+xb+yc+yxk)$. Suppose $X\neq 0$. Then $z=YX^{-1}\in N$ a contradiction to the choice of z. Hence $X=0$ and consequently $Y=0$. Since the set $\{1, x, y, yx\}$ is independent over K, we obtain that $a=b=c=k=e=f=g=h=0$. Thus T is an independent set over K. This completes the proof of the theorem.

The question of existence of infinite proper MVW systems of dimension d over their kerns for $d\geqq 8$ still remains unresolved.

References

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