WHEN PROPER MAPS ARE CLOSED[1]

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A subset F of a space Y is called *compactly closed* if its intersection with each compact subset of Y is compact. A Hausdorff space is called a *k-space* if every compactly closed set is closed. A map of a space X to a space Y is called *proper* if the inverse image of each compact subset of Y is a compact subset of X. The following Lemma is a trivial consequence of these definitions.[2]

LEMMA. *A subset F of a space Y is compactly closed if and only if the inclusion map of the subspace F into Y is proper.*

It is frequently of interest in applications to know that a given proper map is closed. We provide here a simple proof of a criterion due to G. T. Whyburn [1].[3]

THEOREM. *The following four properties of a space Y are equivalent:*
(a) *Every surjective proper map $g: Z \to Y$ is closed.*
(b) *Every proper map $f: X \to Y$ is closed.*
(c) *Every injective proper map $h: X \to Y$ is closed.*
(d) *Every compactly closed subset F of Y is closed.*

PROOF. Assuming (a) and given a proper map $f: X \to Y$, let Z denote the topological sum of X and Y and define $g: Z \to Y$ by $g | X = f$ and $g | Y = i_Y$. The g is clearly surjective and proper, hence closed, and since X is closed in Z, f is closed. Hence (a)$\Rightarrow$(b). Trivially (b)$\Rightarrow$(c), while (c)$\Rightarrow$(d) is immediate from the Lemma. Since (b)$\Rightarrow$(a) is trivial it remains to deduce (b) from (d). Let $f: X \to Y$ be a proper map and F a closed subset of X. We must show $f(F)$ is closed in Y, and since we are assuming (d), it suffices to show that if K is a compact subset of Y then $K \cap f(F)$ is compact. But $K \cap f(F) = f(f^{-1}(K) \cap F)$.

Since f is proper, $f^{-1}(K)$ and hence $f^{-1}(K) \cap F$ are compact and hence so is $f(f^{-1}(K) \cap F)$.   Q.E.D.

It is well known and trivial that a Hausdorff space satisfying the first axiom of countability is a k-space, hence:

COROLLARY. *If Y is a metrizable space or, more generally, a first-countable Hausdorff space or, more generally still, a k-space then every proper map into Y is closed.*

REFERENCE

1. G. T. Whyburn, *Directed families of sets and closedness of functions*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 688–692. MR **32** #435.

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