A SHORT PROOF OF CURRY'S NORMAL FORM THEOREM

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In Chapter 6 of their book [**1**] Curry and Feys define a notion of reduction (strong reduction) for the extensional theory of equality in combinatory logic, show [**1**, Theorem 3, p. 221] that strong reduction has the Church-Rosser property, and define a notion of normal form in analogy with the corresponding concept in lambda-conversion. Curry's normal form theorem [**1**, Theorem 7, p. 230] asserts that if a term ("ob") of combinatory logic is in normal form, it is irreducible, so that if X has normal form X^*, then X reduces to X^* by a process (namely, strong reduction) that cannot be continued further.

Curry's proof of his theorem in [**1**] is quite long and difficult (see [**3**, p. 228] for comment). There is another lengthy proof in the current draft of [**2**] and the first author has discovered a proof using his axiomatization of strong reduction [**3**]. The present proof follows the same general line as the latter proof, but it is considerably shorter and simpler.

Definitions and notation are as in [**3**], except for the symbol $\triangleright$ which is used here for weak reduction.

1. Substitution and abstraction.

The first result is essentially from [**4**, Lemmas 1 and 4]. The proofs, by induction, are easy.

LEMMA 1. *Let P be a redex scheme. Then:*

(a) *P contains at most one occurrence of each meta-variable;*

(b) *If M is a meta-variable occurring in P, there is an N such that NM occurs in P;*

(c) *If P is not basic (i.e. is the result of at least one application of scheme* (viii) *of* [**3**, p. 233]), *then P is weakly irreducible;*

(d) *If P is not basic, then either $P \equiv \mathbf{S}P_1P_2$ or $P \equiv \mathbf{S}P_1$ where P_1 contains at least one occurrence of an atomic combinator.*

The hypotheses of the next lemma are, in essence, the properties of redex schemes asserted in Lemma 1(a), (b), and (c).

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LEMMA 2. *Suppose that P is weakly irreducible, that P contains at most one occurrence of each meta-variable, and that either P is itself a meta-variable, or the occurrence of any meta-variable M in P is in a component NM of P. If $[X/A][Y/B][Z/C]P \rhd R$, then $R \equiv [X^\circ/A] \cdot [Y^\circ/B][Z^\circ/C]P$, where $X \rhd X^\circ$, $Y \rhd Y^\circ$, and $Z \rhd Z^\circ$.*

PROOF. The proof is by induction on P, with three basic clauses: $P \equiv M$; $P \equiv NM$, where N does not contain any meta-variables; and P does not contain any meta-variables. The induction step, $P \equiv P_1P_2$, uses the fact that no substitution instance of P is itself a weak redex.

The next result is Lemma 11 of [3].

LEMMA 3. *Let P be a redex scheme and suppose that U, V, W do not contain x. Then $[U/A][V/B][W/C][x]P_{ijk} \equiv [x][U/A][V/B] \cdot [W/C]P_{ijk} \equiv [x][U^i/A][V^j/B][W^k/C]P$.*

The final preliminary result is an immediate consequence of the Church-Rosser Theorem for weak reduction (see, e.g. [2], or [5, Theorem 12]). It can also be proved directly by induction on Y.

LEMMA 4. *If Y is weakly irreducible, $X \equiv [x]Y$, and $Xx \rhd Q$, then $Q \rhd Y$.*

2. **The normal form theorem.** By [3], X is irreducible if it contains no redexes, so Curry's theorem may be stated as follows.

NORMAL FORM THEOREM. *If X is in normal form, it contains no redexes.*

PROOF. The proof is by induction on the definition of normal form.

(1) If $X \equiv x$, then X contains no redex by Lemma 1(b) or (d).

(2) If $X \equiv xX_1 \cdots X_n$, with $X_1, \cdots, X_n$ in normal form, assume by induction that $X_1, \cdots, X_n$ contain no redexes. The only possible redexes in X then have x at the head, which is impossible by Lemma 1(d).

(3) If $X \equiv [x]Y$, where Y is in normal form, assume by induction that Y contains no redexes. The rest of the proof is by induction on Y. Note that Y is weakly irreducible.

(3a) If $Y \equiv x$, then $X \equiv \mathbf{I}$, which is not a redex by Lemma 1(d).

(3b) If Y does not contain x, then $X \equiv \mathbf{K}Y$. By Lemma 1(d), neither X nor $\mathbf{K}$ is a redex, so any redexes in X must be components of Y.

(3c) If $Y \equiv Xx$ and contains no redexes, clearly X contains none.

(3d) If $Y \equiv Y_1Y_2$ and $X \equiv \mathbf{S}X_1X_2$ where $X_i \equiv [x]Y_i$, $i = 1, 2$, then assume the theorem for X_1 and X_2. Then the only possible redexes in

X (by Lemma 1(d)) are $\mathbf{S}X_1$ and X itself. Consider the two cases separately.

(3d1) Suppose $\mathbf{S}X_1$ is a redex. Then $\mathbf{S}X_1$ is a substitution instance of a redex scheme $\mathbf{S}R$. If $\mathbf{S}R$ is basic, then $\mathbf{S}R \equiv \mathbf{S}(\mathbf{KI})$, so $\mathbf{KI} \equiv [x]Y_1$. Hence $Y_1 \equiv \mathbf{I}$, contradicting the hypothesis that Y is weakly irreducible. Thus, $\mathbf{S}R \equiv [x]P_{ijk}$ for a redex scheme P. This implies that $P \equiv \mathbf{S}RM$ for a meta-variable M, so that X itself is also a redex. Hence, this case may be included in the next.

(3d2) Suppose $X \equiv \mathbf{S}X_1X_2$ is a redex, a substitution instance of a redex scheme $\mathbf{S}R_1R_2$. If $\mathbf{S}R_1R_2$ is basic, then $R_1 \equiv \mathbf{K}A$, so $X_1 \equiv \mathbf{K}U \equiv [x]Y_1$. Then $Y_1 \equiv U$ and does not contain x. Either $R_2 \equiv \mathbf{I}$ or $R_2 \equiv \mathbf{K}B$. In the first case, $X_2 \equiv \mathbf{I}$, so $Y_2 \equiv x$. But then $X \equiv [x]Ux \equiv U$, which is impossible. In the second case, $X_2 \equiv \mathbf{K}V$ and $Y_2 \equiv V$ so that Y does not contain x, and $X \equiv [x]Y \equiv \mathbf{K}Y$, contrary to hypothesis. Thus, $\mathbf{S}R_1R_2$ is not basic.

We may now suppose that $\mathbf{S}R_1R_2 \equiv [x]P_{ijk}$ for a redex scheme P, so that $X \equiv [U/A][V/B][W/C][x]P_{ijk}$. Since $X \equiv [x]Y$, X does not contain x, and hence, neither do U, V, W. Thus, Lemma 3 applies and $X \equiv [x]Q$, where $Q \equiv [U^i/A][V^j/B][W^k/C]P$. Moreover, we must have $Q \equiv Q_1Q_2$ and $X \equiv \mathbf{S}([x]Q_1)([x]Q_2)$, for the alternative is that $Q \equiv \mathbf{S}U^1V^1x \equiv \mathbf{S}UVx$, so that $\mathbf{S}R_1R_2 \equiv \mathbf{S}AB$, which is not a redex scheme. Then by Lemma 4, $Q \rhd Y$, $Q_1 \rhd Y_1$, and $Q_2 \rhd Y_2$.

If P is weakly irreducible, then Lemma 2 (with $Y \equiv R$) implies that $Y \equiv [U^{i\circ}/A][V^{j\circ}/B][W^{k\circ}/C]P$. This contradicts the hypothesis that Y is not a redex.

If P is weakly reducible, then either $P \equiv \mathbf{S}ABC$ or $P \equiv \mathbf{K}AB$, so that either $Q \equiv \mathbf{S}U^iV^jW^k$ or $Q \equiv \mathbf{K}U^iV^j$. But since $Q_1 \rhd Y_1$, either $\mathbf{S}U^iV^j \rhd Y_1$ and hence $Y_1 \equiv \mathbf{S}Y'Y''$, or else $\mathbf{K}U^i \rhd Y_1$, and hence $Y_1 \equiv \mathbf{K}Y'$. In either case, Y is not weakly irreducible. This final contradiction completes the proof.

References

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