A CLOSED SUBSPACE OF $\mathfrak{D}(\Omega)$ WHICH IS NOT AN LF-SPACE

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ABSTRACT. With proper choice of region $\Omega \subseteq R^n$ and constant coefficient linear partial differential operator P, namely Ω being P-convex but not strong P-convex, the range of P in $\mathfrak{D}(\Omega)$ is a closed subspace of $\mathfrak{D}(\Omega)$ whose subspace topology differs from its canonical LF-topology. In the present paper this result is proved and an example of a pair Ω, P satisfying the above hypotheses is presented.

Dieudonné and Schwartz, in their pioneering work [1] on locally convex spaces, raised a number of fundamental questions concerning LF-spaces. X is an LF-*space* if $X = \bigcup X_n$ where $\{X_n\}$ is a strictly increasing sequence of Fréchet spaces such that the canonical injection of X_n into X_{n+1} is a homeomorphism for every n and X is equipped with the finest locally convex topology making the canonical injection of X_n into X continuous for every n. One of the questions was:

If M is a closed subspace of X, is M an LF-space?

Grothendieck [2, p. 89] gave an ingenious example involving Köthe spaces which answered this question (together with many others) negatively. We here present another example of a closed subspace of an LF-space which is not an LF-space. However, this example involves the principal LF-spaces arising in the theory of distributions, namely the class of spaces $\mathfrak{D}(\Omega)$ where Ω is an open subset of $\boldsymbol{R}^n$, and the subspace of $\mathfrak{D}(\Omega)$ exhibiting the pathology is the image of $\mathfrak{D}(\Omega)$ under a linear partial differential operator with constant coefficients. We are indebted to Richard M. Aron for clarifying our results.

Hörmander proved [3, Theorems 3.63 and 3.64, pp. 85–87] that $P(D)\mathfrak{D}'(\Omega) = \mathfrak{D}'(\Omega)$ if and only if Ω is strongly P-convex. This powerful theorem provides the key to proving the following result.

THEOREM. *If Ω is P-convex, but not strongly P-convex, then $P(-D)\mathfrak{D}(\Omega)$ is a closed subspace of $\mathfrak{D}(\Omega)$ which is not an LF-space.*

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PROOF. It is shown in Trèves [**5**, Theorem 5.1, p. 287] that if Ω is P-convex, then $P(-D)\mathfrak{D}(\Omega)$ is a closed subspace of $\mathfrak{D}(\Omega)$. Now suppose $P(-D)\mathfrak{D}(\Omega)$ is an LF-space. By the open mapping theorem (see Husain [**4**, p. 44]), $P(-D):\mathfrak{D}(\Omega)\to P(-D)\mathfrak{D}(\Omega)$ is a linear homeomorphism. Let $T\in\mathfrak{D}'(\Omega)$. Then the linear form $P(-D)f\to T(f)$ is continuous and hence by the Hahn Banach theorem can be extended to a continuous linear functional S on $\mathfrak{D}(\Omega)$. Since $S(P(-D)f)=T(f)$ for all $f\in\mathfrak{D}(\Omega)$, we conclude that $P(D)S=T$. Therefore $P(D)\mathfrak{D}'(\Omega)=\mathfrak{D}'(\Omega)$, contradicting the fact that Ω is not strongly P-convex. Hence $P(-D)\mathfrak{D}(\Omega)$ is not an LF-space.

For the sake of completeness, we explicitly provide an example of a constant coefficient linear partial differential operator and an open set in $\boldsymbol{R}^3$ satisfying the hypothesis of the theorem.

EXAMPLE. Let $\Omega=\{x\in\boldsymbol{R}^3:\phi(x)=x_1^2-x_2^2-x_3^2-1<0\}$, and $P(D)=D_1^2+D_2^2-D_3^2$. Then Ω and P satisfy the hypothesis of the theorem. First, there are no points on $\mathrm{bd}(\Omega)$ which are characteristic with respect to P. The P-convexity of Ω follows directly from a result of Hörmander [**3**, Theorem 3.74, p. 92]. However, Ω fails to be strongly P-convex because the normal curvature at the point (1, 0, 0) in the tangential bicharacteristic direction (0, 1, 1) is negative (see Hörmander [**3**, Theorem 3.75, p. 93]).

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