A DIRECT PROOF THAT A LINEARLY ORDERED SPACE IS HEREDITARILY COLLECTIONWISE NORMAL[1]

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Although it appears well known that a linearly ordered space is completely normal (=hereditarily normal), most available proofs (in, for instance, [1] and [2]) are very indirect. In this paper we present a direct proof of a stronger theorem, namely that the interval topology is hereditarily collectionwise normal.[2]

If X is linearly ordered, we will call a set $S \subset X$ *convex* if $a, b \in S$ and $a < t < b$ implies $t \in S$. The union of any collection of convex sets with nonempty intersection is convex, so any subset S of X can be uniquely expressed as a union of disjoint maximal convex sets called *convex components*. Clearly every interval in X is convex but not conversely, and we will, as usual, denote intervals by (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. In what follows, X will denote a *linearly ordered space*, i.e., a linearly ordered set endowed with the usual open interval topology.

Suppose $\{A_i\}$ is a discrete family of subsets of X. Let

$$A_i^* = \bigcup \{ [a, b] \mid a, b \in A_i, \ [a, b] \cap A_j = \emptyset \ \ \forall j \neq i \}.$$

Then $A_i \subset A_i^*$, and $A_i^* \cap A_j^* = \emptyset$ whenever $i \neq j$; in fact, the family $\{A_i^*\}$ is discrete. To prove this, we select for each $x \in X$ a neighborhood I_x which intersects at most one of the sets A_i. If I_x meets exactly one element of $\{A_i\}$, say A_k, and if x is not an endpoint of X, we can take I_x to be an interval (s, t). Then if $i \neq k$, (s, t) may intersect A_i^* only if it intersects some interval $[a, b]$ where $a, b \in A_i$. But since $(s, t) \cap A_i = \emptyset$ and $a, b \in A_i$ then $(s, t) \subset (a, b)$ which would imply that $A_k \cap A_i^* \neq \emptyset$. But this is impossible if $i \neq k$, so in this case I_x can intersect at most one of the sets A_i^*. Other cases are treated analogously, so $\{A_i^*\}$ (and consequently $\mathrm{cl}(A_i^*)$) is discrete.

If we now write each A_i^* and $(\bigcup_i A_i^*)'$ as the union of convex components, $A_i^* = \bigcup_\alpha A_\alpha^i$, and $(\bigcup_i A_i^*)' = \bigcup_\gamma C_\gamma$, the collection $M = \{A_\alpha^i, C_\gamma\}$ inherits a linear order from X and is thus itself a linearly ordered set. We claim that in the ordered set M, each of the sets A_α^i has an immediate successor whenever A_α^i intersects the closure of S_α^i, the set of strict upper bounds for A_α^i. For suppose $A_\alpha^i \cap S_\alpha^i \neq \emptyset$. Then $A_\alpha^i \cap \mathrm{cl}(S_\alpha^i)$

Received by the editors December 2, 1968.

[1] This work was partially supported by the Research Corporation.

[2] The author wishes to thank the referee for several clarifying suggestions.

contains precisely one point, say p, every neighborhood of which intersects A_i^*. Thus since $\mathrm{cl}(A_i^*)$ is discrete, there exists a neighborhood (x, y) of p disjoint from $\bigcup_{j\neq i} \mathrm{cl}(A_j^*)$. Then $(x, y)\cap S_\alpha^i \neq \emptyset$, so $(p, y)\neq\emptyset$. But the definition of A_i^* insures that (p, y) is disjoint from both A_i^* and $\bigcup_{i\neq j} A_j^*$, so there must exist some set C_γ containing (p, y). In the linear order on M, C_γ is the immediate successor to A_α^i, and we will call it C_α^{i+}.

For each γ, select and fix some point $k_\gamma\in C_\gamma$. Then whenever $A_\alpha^i\cap\mathrm{cl}(S_\alpha^i\neq\emptyset$, there exists a unique $k_\alpha^{i+}\in C_\alpha^{i+}$, the immediate successor of A_α^i. In such cases, let $I_\alpha=[p, k_\alpha^{i+})$ where $p\in A_\alpha^i\cap\mathrm{cl}(S_\alpha^i)$; otherwise, if $A_\alpha^i\cap\mathrm{cl}(S_\alpha^i=\emptyset$, let $I_\alpha^i=\emptyset$. Define J_α^i similarly for the strict lower bounds of A_α^i (using the same collection of points $k_\gamma\in C_\gamma$). Then for each α and each i, let $U_\alpha^i=J_\alpha^i\cup A_\alpha^i\cup I_\alpha^i$. Each U_α^i is clearly an open set containing A_α^i, so $U_i=\bigcup_\alpha U_\alpha^i$ is an open set containing A_i^*. Since no A_α^i intersects any A_β^j for $i\neq j$, and since the use of the same k_γ throughout implies that no J_α^i or I_α^i may intersect any J_β^j or I_β^j, it is clear that no U_α^i can intersect any U_β^j for $i\neq j$. Thus $U_i\cap U_j=\emptyset$ whenever $i\neq j$, and hence X is collectionwise normal.

Now every subspace of X inherits both a topology as well as a linear order; these need not be compatible, even for open subspaces. (The open subspace $\{\alpha+1 \mid \alpha \text{ is a limit ordinal}\}$ of the linearly ordered ordinal space $\{\gamma \mid \gamma<\Omega\}$ inherits the discrete topology but is of the same order type as the countable ordinals.) However, the two structures are compatible on convex subspaces of X, whence convex subspaces of X are collectionwise normal. Therefore any open subset of X —being the disjoint union of open collectionwise normal subspaces (namely its convex components)—is collectionwise normal. This suffices to prove that every subset S of X is collectionwise normal, since if $\{A_i\}$ is a discrete family in S, then each point $s\in S$ has a neighborhood $U_s\cap S$ which meets at most one of the sets A_i. But then $U=\bigcup_s U_s$ is an open set with the same property, and since U is collectionwise normal, so must be S. Hence X is hereditarily collectionwise normal.

That X is completely normal (i.e., hereditarily normal) follows as a corollary. But it also may be proved more directly by a slight modification of the proof that X is collectionwise normal.

References

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