A RANDOM L^1 FUNCTION WITH DIVERGENT WALSH SERIES

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ABSTRACT. The purpose of this paper is to point out that the techniques of J. P. Kahane to arrive at almost everywhere divergent Fourier series may be carried over to the Fourier-Walsh system. In particular we construct a random L^1 function whose Fourier-Walsh series almost surely (a. s.) diverges almost everywhere.

1. As an application of his theorem on operations of weak type E. M. Stein [4] has proven the existence of a L^1 function whose Fourier-Walsh series is divergent almost everywhere. Recently Billard [1] has shown in direct analogy with Fourier series that the Fourier-Walsh series of L^2 functions are a. e. convergent. Our reference for Fourier-Walsh series is the paper of Fine [2].

Following Kahane [3, pp. 97–114], we begin with a sequence of positive numbers $m_1, m_2, \cdots, m_j, \cdots$ such that $\sum m_j < \infty$ and form the random measure $\sum m_j \delta_{\theta_j} = d\mu$ where $\{\theta_j\}$ is an independent sequence of random points equidistributed on $[0, 1]$, and δ_{θ_j} is the unit point mass measure at θ_j. It turns out, just as with Fourier-Stieltjes series of $d\mu$, that the Fourier-Walsh-Stieltjes series of $d\mu$ is a. s. bounded at almost every point if $\sum m_j \log(1/m_j) < \infty$ and is a. s. not bounded at almost every point if $\sum m_j \log(1/m_j) = \infty$.

2. **Random measures.** We shall write $S(t; d\mu)$ for the Fourier-Walsh-Stieltjes series of $d\mu$ and $S_n(t; d\mu)$ for the partial sums. Let $D_n(t) = \sum_{j=0}^{n-1} \psi_j(t)$ be the Dirichlet kernel in the Walsh system $\{\psi_j\}$, then $S_n(t; d\mu) = \sum_{j=1}^{\infty} m_j D_n(t \dotplus \theta_j)$. (Addition of reals is dyadic, cf. [2].) For convenience we shall write $D(k, \theta)$ for $D_{2^k}(\theta)$. We shall need the following facts: $\psi_{2^n} = \phi_n$, the nth Rademacher function; $\psi_r(\theta) \cdot \psi_s(\theta) = \psi_{r+s}(\theta)$ provided r and s have dyadic expansions with no common exponents; and finally $D(k, \theta) = 2^k$ on $[0, 2^{-k})$ and is 0 on $[2^{-k}, 1]$.

THEOREM 1. *If* $\sum m_j < \infty$ *and* $\sum m_j \log(1/m_j) < \infty$ *then a. s.* $S(t; d\mu)$ *is bounded almost everywhere.*

PROOF. Since for every t, $t \dotplus \theta_j$ and θ_j have the same distribution

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the series $S(t;\, d\mu)$ and $S(0;\, d\mu)$ are similar. As in [3, p. 99], the hypotheses imply that $\sum(m_j/\theta_j)<\infty$ a. s. Also $|D_k(\theta)|\leqq 2/\theta$ for $0<\theta<1$ and all k [2, p. 391], and the proof is complete.

For the converse we need the following elementary lemma whose proof is omitted.

LEMMA. *Let $\theta_1, \cdots, \theta_v$ be independent random variables equidistributed on $[0, 1]$ and let $\epsilon_1, \cdots, \epsilon_v$ be an arbitrary assignment of signs, then for every integer N there is a. s. an $n_0>N$ such that $\phi_{n_0}(\theta_i)=\epsilon_i$ for $1\leqq i\leqq v$.*

THEOREM 2. *If $\sum m_j<\infty$ and $\sum m_j \log(1/m_j)=\infty$ then a. s. $\sup_n S_n(t;\, d\mu)=\infty$ almost everywhere.*

PROOF. As in [3] the hypotheses imply that a. s. $\sum(m_j/\theta_j)=\infty$. That for each v and n $\mathrm{p}(\sum_{j=v+1}^{\infty} m_jD_n(\theta_j)\geqq 0)\geqq 1/2$ is an immediate consequence of the fact that if $m\geqq 1$ $\psi_m(\theta)$ is a symmetric random variable. To see this suppose $\psi_m=\phi_{m_1}\cdots\phi_{m_k}$ where $m_1>m_2>\cdots>m_k$. Then

$$\begin{aligned}\psi_m(\theta\dotplus 2^{-(m_k+1)}) &= \phi_{m_1}(\theta\dotplus 2^{-(m_k+1)})\cdots\phi_{m_k}(\theta\dotplus 2^{-(m_k+1)})\\ &= \phi_{m_1}(\theta)\cdots\phi_{m_{k-1}}(\theta)\phi_{m_k}(\theta\dotplus 2^{-(m_k+1)})\\ &= -\phi_{m_1}(\theta)\cdots\phi_{m_k}(\theta) = -\psi_m(\theta).\end{aligned}$$

Thus by the zero-one law it suffices to show that

$$\sup_n \sum_{j=1}^{v} m_jD_n(\theta_j)\geqq\left(\frac{1}{2}\right)^3\sum_{j=1}^{v}\frac{m_j}{\theta_j}$$

a. s. for each v.

If $n_0>n_1>\cdots>n_q$ we have

$$D_w = D(n_0)+\phi_{n_0}D(n_1)+\phi_{n_0}\phi_{n_1}D(n_2)+\cdots\phi_{n_0}\phi_{n_1}\cdots\phi_{n_{q-1}}D(n_q)$$

where $w=2^{n_0}+2^{n_1}+\cdots+2^{n_q}$.

Almost surely none of $\theta_1, \cdots, \theta_v$ are dyadic rationals. Assume they are contained in $\bigcup_{i=1}^{q} K_i$ where $K_i=\{\theta\,|\,2^{-n_{i-1}}<\theta<2^{-n_i}\}$ where $n_i=n_1-2(i-1)$, $i=1, 2, \cdots, q$. Now by the lemma we may a. s. choose $n_0>n_1$ such that $\phi_{n_0}(\theta_i)=1$ if $\theta_i\in K_1$ and $\phi_{n_0}\phi_{n_1}(\theta_i)=1$ if $\theta_i\in K_2$ and $\cdots\phi_{n_0}\phi_{n_1}\cdots\phi_{n_{q-1}}(\theta_i)=1$ if $\theta_i\in K_q$.

Now if $\theta_k\in K_i$ we have that $D_w(\theta_k)>2^{n_i}-\sum_{j=0}^{n_i+1}2^j=2^{n_i-1}+1$. Also $1/\theta_k<2^{n_{i-1}}$ so that $1/2^3\theta_k<2^{n_i-1}<D_w(\theta_k)$. Hence a. s.

$$\sum_{j=1}^{v} m_jD_w(\theta_j)\geqq\left(\frac{1}{2}\right)^3\sum_{j=1}^{v}\frac{m_j}{\theta_j},$$

and the proof is complete.

REMARK. Contrary to the situation with Fourier-Stieltjes series, it is false here that a. s. $\sup_{n\in\Lambda} S_n(t;\, d\mu) = \infty$ a. e. for Λ an infinite set of positive integers. For example if $\Lambda = \{2^n\}$ then $S_{2^n}(0;\, d\mu) = \sum_j m_j D(n,\, \theta_j)$ is a. s. bounded. To see this let $E_j =$ the event $\theta_j \notin [2^{-j},\, 1]$ then $\sum \mathrm{p}(E_j) < \infty$, so by the Borel-Cantelli lemma $\theta_j \in [2^{-j},\, 1]$ for all but finitely many j with probability 1. Thus a. s. $\sup_n S_{2^n}(0;\, d\mu) < \infty$ and hence a. s. $\sup_n S_{2^n}(t;\, d\mu) < \infty$ a. e. (The referee has kindly pointed out that this is also a consequence of the classical fact that the partial sums $S_{2^n}(t;\, d\mu)$ are bounded in $L^1(0,\, 1)$.)

3. **A random L^1 function with divergent Walsh series.** In this section we assume $\sum m_j < \infty$ and $\sum m_j \log(1/m_j) = \infty$. Since a. s. $\lim_n \sup S_n(0;\, d\mu) = \infty$, a positive sequence $\{a_n\}$, $a_n \searrow 0$, may be found such that a.s. $\lim_n \sup a_n S_n(0;\, d\mu) = \infty$ and hence such that a. s. $\lim_n \sup a_n S_n(t;\, d\mu) = \infty$ a. e. Furthermore choose $a_0 = a_1$, $a_{2^n+k} = a_{2^n}$ for $0 \leqq k < 2^n$ and $n = 1,\, 2,\, \cdots$.

Now define the positive L^1 function $f(t) = \sum_{n=0}^{\infty} (a_{2^n-1} - a_{2^n}) D(n,\, t)$. Since $\|D(n)\|_1 = 1$ it follows that the series is absolutely L^1-convergent. The jth Walsh coefficient of f is $\hat{f}(j) = \sum_{2^n > j} (a_{2^n-1} - a_{2^n}) \hat{D}(n,\, j) = a_{2^m-1} = a_j$ where $2^{m-1} \leqq j < 2^m$.

We now form the random L^1 function $f * d\mu(t) = \sum_{j=1}^{\infty} m_j f(t \dot{+} \theta_j)$ and note that $S_n(t;\, f * d\mu) = a_n S_n(t;\, d\mu) + \sum_{m=0}^{n} C_m (a_m - a_n) \psi_m(t)$ where C_m denotes the mth Fourier-Walsh-Stieltjes coefficient of $d\mu$.

The last term is $f_n * d\mu$ where $f_n = \sum_{m=0}^{n} (a_m - a_n) \psi_m$. To see that f_n is positive set $b_k = a_k - a_n$ and note that f_n has the form

$$\sum_{k=0}^{2^r-1} b_k \psi_k = b_0 D(1) + \sum_{k=1}^{r-1} b_{2^k} (D(k+1) - D(k))$$

$$= \sum_{k=1}^{r-1} (b_{2^{k-1}} - b_{2^k}) D(k) + b_{2^{r-1}} D(r) \geqq 0.$$

Thus we see that $S_n(t;\, f * d\mu) \geqq a_n S_n(t;\, d\mu)$ and it follows that a.s. $\sup_n S_n(t; f * d\mu) = \infty$ a.e.

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