ON l-l SUMMABILITY

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1. **Introduction.** Let (E, p_i) and (F, q_j) be Fréchet spaces, i.e., locally convex Hausdorff spaces which are metrisable and complete, whose topologies are generated, respectively, by the countable collections $\{p_i\}$ and $\{q_j\}$ of seminorms. Let the infinite matrix $A \equiv (A_{nk})$ consist of entries A_{nk} each of which is a continuous linear operator of E into F. Given a sequence $\{x_k\}$ in E we (formally) define a sequence $\{y_n\}$ by

$$y_n = \sum_{k=0}^{\infty} A_{nk}x_k, \qquad n = 0, 1, 2, \cdots. \tag{1.1}$$

We say the matrix A is an l-l method if each series (1.1) converges in (F, q_j) and

$$\sum_{n=0}^{\infty} q_j(y_n) < +\infty, \qquad j = 1, 2, \cdots,$$

whenever

$$\sum_{k=0}^{\infty} p_i(x_k) < +\infty, \qquad i = 1, 2, \cdots.$$

We say the method A is absolutely L-regular if in addition $\sum_{n=0}^{\infty} y_n = \sum_{k=0}^{\infty} L(x_k)$ whenever $\sum_{k=0}^{\infty} p_i(x_k) < +\infty$, $i = 1, 2, \cdots$. Here L is a prescribed continuous linear operator of E into F. It is the purpose of this note to establish necessary and sufficient conditions which ensure that A be l-l or absolutely L-regular. For the classical case (E, F the complex numbers with the usual topology) these conditions were given by Mears [**3**] and Knopp and Lorentz [**1**] and for the Banach space setting by Lorentz and Macphail [**2**].

2. **Theorems.**

THEOREM 2.1. *The matrix $A = (A_{nk})$ defining series to series transformations from the F-space (E, p_i) into the F-space (F, q_j) is l-l if and only if*

(2.1) *for each bounded set M_α in E and for each fixed j,*

$$\sum_{n=0}^{m} q_j(A_{nv}(x_v)) \leqq K_{\alpha,j} \qquad \text{for } m, v = 0, 1, 2, \cdots$$

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and $x_v \in M_\alpha$, $v=0, 1, 2, \cdots$.

The proof of Theorem 2.1 requires the following lemmas. Lemma 2.2 is known [4], while Lemma 2.3 is a minor modification of Lemma 2.4 of [5].

LEMMA 2.2. *If E and F are locally convex spaces and E is quasicomplete then any collection of continuous linear operators from E into F which is simply bounded is bounded for the topology of uniform convergence on bounded sets.*

LEMMA 2.3. *If* $\sum_{k=0}^{\infty} A_{nk}x_k$ *converges in F for every sequence* $\{x_k\}$ *in E such that* $\sum_{k=0}^{\infty} p_i(x_k)$ *converges* ($i=1, 2, \cdots$), *then the sequence* $\{A_{nk}\}$, $k=0, 1, \cdots$, *of continuous linear operators from E into F is bounded (for fixed n) for the topology of uniform convergence on bounded sets.*

PROOF OF THEOREM 2.1. Assume $A=(A_{nk})$ is *l-l* and consider the linear space E_1 of sequences $\{x_k\}$ in E such that $\sum_{k=0}^{\infty} p_i(x_k) < +\infty$ ($i=1, 2, \cdots$). For $x=\{x_k\}$ in E_1 define $P_i(x)=\sum_{k=0}^{\infty} p_i(x_k)$. Then, for each i, P_i is a seminorm on E_1 and the locally convex space $(E_1 P_i)$ is complete. Now, since $A=(A_{nk})$ is *l-l*, each series $\sum_{k=0}^{\infty} A_{nk}(x_k)$ converges in (F, q_j) whenever $\sum_{k=0}^{\infty} p_i(x_k) < +\infty$, $\{x_k\}$ in E. It follows from Lemma 2.3 that $\{A_{nk}\}$, $k=0, 1, \cdots$, is bounded for the topology of uniform convergence on bounded sets. We shall show that this implies

(2.2) for each $n=0, 1, 2, \cdots$, $i=1, 2, \cdots$ and $j=1, 2, \cdots$, there exists a number $K_{n,j,i} \geqq 0$ such that $q_j(A_{nk}(x)) \leqq K_{n,j,i} p_i(x)$, $x \in E$, $k=0, 1, 2, \cdots$.

For each $k=0, 1, \cdots$ and fixed i, j, n, define $\mu_k(x)=q_j(A_{nk}(x))$, $x \in E$. Then μ_k is a seminorm on E. It follows from the fact that $\{A_{nk}\}$ ($k=0, 1, \cdots$) is bounded for the topology of uniform convergence on bounded sets that there exists a number, $K_{n,j,i} \geqq 0$, such that $p_i(x) < 1$ and $k=0, 1, \cdots$ imply $\mu_k(x) \leqq K_{n,j,i}$. If $K_{n,j,i} > 0$ it is easy to see that (2.2) holds. On the other hand, if $K_{n,j,i}=0$ (2.2) follows from elementary properties of seminorms (see, e.g., the proof of Theorem 2.1 in [5]). Thus for each fixed $n=0, 1, \cdots$ the linear operator T_n defined by $T_n(x)=\sum_{k=0}^{\infty} A_{nk}(x_k)$, $x=\{x_k\} \in E_1$, is in $L(E_1, F)$, i.e., T_n is a continuous linear operator from E_1 into F. Let F_1 denote the linear space of sequences $\{y_k\}$ in F such that $\sum_{k=0}^{\infty} q_j(x_k) < +\infty$. For $y=\{y_k\} \in F_1$ define $Q_j(y)=\sum_{k=0}^{\infty} q_j(y_k)$. Then (F_1, Q_j) is a locally convex complete seminormed space. Define the operators U_m, $m=0, 1, \cdots$, by $U_m(x)=\{y_n\}$ where

$$y_n = T_n(x), \qquad n = 0, 1, \cdots, m,$$
$$= 0, \qquad n > m,$$

and $x \in E_1$. Thus $U_m \in L(E_1, F_1)$, $m = 0, 1, \cdots$. Since A is l-l, $\{U_m(x)\}$ converges in (F_i, Q_j) whenever $x \in E_1$. It now follows from Lemma 2.2 that U_m, $m = 0, 1, \cdots$, is bounded for bounded convergence on $L(E_1, F_1)$. Therefore, for each fixed j and each bounded set M in E_1

$$\sup_{x \in M} Q_j(U_m(x)) \leqq K_{M,j}, \qquad m = 0, 1, \cdots.$$

Consider a bounded set M_α in E. Say M_α consists of points x such that $p_i(x) < \alpha_i$. Consider sequences of the form $\{x_0, 0, 0, \cdots\}$, $\{0, x_1, 0, \cdots\}$, $\{0, 0, x_2 0, \cdots\} \cdots$, where the x_i's are in M_α. All such sequences are in the same bounded set U_α of (E_1, P_i). Therefore,

$$\begin{aligned} Q_j(U_m(x)) &= Q_j(\{T_0(x), T_1(x), \cdots, T_m(x), 0, 0, 0, \cdots\}) \\ &= \sum_{n=0}^{m} q_j(T_n(x)) = \sum_{n=0}^{m} q_j(A_{nv}(x_v)) \\ &\leqq K_{m_\alpha,j} = K_{\alpha,j}, \end{aligned}$$

for $m, v = 0, 1, 2, \cdots$ and $x_v \in M_\alpha$ for each v, i.e., (2.1) holds.

Conversely, suppose (2.1) is true. Let $x = x_k \in E_1$. Then $x_k \in$ (same) bounded set M_α in E for $k = 0, 1, \cdots$. For each j and each $n = 0, 1, \cdots$, we claim that

$$\sum_{k=0}^{\infty} q_j(A_{nk}(x_k)) < +\infty. \tag{2.3}$$

For, by (2.1), given j there exists $K_{\alpha,j} \geqq 0$ such that $q_j(A_{nk}(x_k)) \leqq K_{\alpha,j}$, $n, k = 0, 1, \cdots$. It now follows, as in the first part of the proof, that there exists a number $R = R(n, j, i)$ such that $q_j(A_{nk}(x_k)) \leqq R p_i(x_k)$ for $k = 0, 1, \cdots$. Thus (2.3) is valid.

Since

$$y_m = \sum_{n=0}^{m} q_j\left(\sum_{k=0}^{\infty} A_{nk}(x_k)\right)$$

is a nondecreasing sequence of nonnegative numbers, it suffices to show that $\{y_m\}$ is bounded above in order to conclude that A is l-l. For a given j, $x \in E$ and $v = 0, 1, \cdots$, define

$$S_j(x) = \sum_{n=0}^{\infty} q_j(A_{nv}(x)).$$

It follows, using (2.1), that S_j is a seminorm on E. Property (2.1) now implies, as before, the existence of a number $T \geqq 0$, where T depends on j, i but is independent of n and v ($n, v = 0, 1, \cdots$), such that $S_j(x) \leqq Tp_i(x)$ for $x \in E$. Using (2.3) we obtain easily that $\{y_m\}$ is bounded above and the proof is complete.

Under (2.1) we have

$$\lim_m \sum_{n=0}^{m} \sum_{k=0}^{\infty} A_{nk}(x_k) = \lim_m \sum_{k=0}^{\infty} \sum_{n=0}^{m} A_{nk}(x_k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{nk}(x_k).$$

The following theorem is now obvious.

THEOREM 2.4. *The method $A = (A_{nk})$, considered as an l-l method, is absolutely L-regular if and only if* (2.1) *holds and also*

$$\lim_m \sum_{n=0}^{m} A_{nk}(x_k) = L(x_k), \qquad k = 0, 1, \cdots,$$

for $\{x_k\} \in E_1$.

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