SYMMETRY OF GENERALIZED GROUP ALGEBRAS

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In this note we shall consider the generalized group algebras $B^p(G, A)$, where G is a compact Hausdorff group, A a Banach-$*$-algebra, and $1 \leqq p < \infty$. These spaces have been studied by Spicer [**6**] and [**7**] and are defined as the spaces of functions $f: G \to A$ for which

$$\left[\int |f(g)|^p dm(g)\right]^{1/p} < \infty.$$

$B^p(G, A)$ is normed by defining

$$|f|_p = \left[\int |f(g)|^p dm(g)\right]^{1/p}$$

and involution is defined as usual: $f^*(g) = f(g^{-1})^*$.

We prove the following

THEOREM. *If G is a compact group and A is a Banach algebra with (continuous) involution, then $B^p(G, A)$ is symmetric if and only if A is symmetric.*

A Banach-$*$-algebra is symmetric if elements $f^* * f$ have nonnegative spectrum. This is the case if and only if hermitian elements have real spectra [**5**].

We first observe that it suffices to show that A is symmetric iff $B^1(G, A)$ is symmetric, because $B^1(G, A)$ symmetric $\Rightarrow B^p(G, A)$ symmetric for any p, $1 \leqq p < \infty$, $\Rightarrow A$ symmetric $\Rightarrow B^1(G, A)$ symmetric. The proof of the first of these implications will be accomplished by showing that if $t \in B^p(G, A) \subseteq B^1(G, A)$ then the spectrum of t in $B^p(G, A)$, $\sigma_p(t)$ equals the spectrum of t in $B^1(G, A)$, $\sigma_1(t)$. Since $B^p(G, A) \subseteq B^1(G, A)$, clearly $\sigma_1(t) \subseteq \sigma_p(t)$. On the other hand, $B^p(G, A)$ is an ideal in $B^1(G, A)$ [**6**]. Recall that $0 \neq \lambda \notin \sigma_1(t)$ iff t/λ has a quasi-inverse, y_λ, say, in $B^1(G, A)$ [**4**, p. 28]. t/λ and y_λ satisfy the relationship

$$t/\lambda + y_\lambda - (t/\lambda) * y_\lambda = 0$$

or

$$y_\lambda = (t/\lambda) * y_\lambda - t/\lambda.$$

Using the fact that $B^p(G, A)$ is an ideal in $B^1(G, A)$ we conclude

Received by the editors June 4, 1969.

that $y_\lambda \in B^p(G, A)$ i.e. $\lambda \notin \sigma_p(t)$. If neither $B^p(G, A)$ nor $B^1(G, A)$ contains an identity $0 \in \sigma_1(t)$ and $0 \in \sigma_p(t)$. On the other hand, if G is a finite group, then obviously $t \in B^p(G, A)$ has an inverse in $B^p(G, A)$ if and only if t has an inverse in $B^1(G, A)$, because $B^1(G, A) = B^p(G, A)$ setwise. Consequently, we have shown

LEMMA 1. *$\sigma_1(t) = \sigma_p(t)$ for all $t \in B^p(G, A)$ and any p, $1 \leqq p < \infty$.*

From this lemma it follows that if $B^1(G, A)$ is symmetric and $t \in B^p(G, A)$, then $-1 \notin \sigma_1(t^*t)$ and therefore $-1 \notin \sigma_p(t^*t)$. This shows that $-t^*t$ is quasi-regular in $B^p(G, A)$ for any $t \in B^p(G, A)$; therefore $B^p(G, A)$ is symmetric if $B^1(G, A)$ is symmetric.

Next we prove the second implication. Suppose $B^p(G, A)$ is symmetric. We show that A is symmetric: Simply embed A in $B^p(G, A)$ by considering the isometric image of A in $B^p(G, A)$. This identification shows immediately that if $B^p(G, A)$ is symmetric, then A is symmetric.

To show that $B^1(G, A)$ is symmetric, provided that A is, is somewhat more complicated. The proof given here depends on the minimal ideal structure of $L^1(G)$ via the identification $B^1(G, A) = L^1(G) \otimes_\gamma A$ [**6**], based on a result by Grothendieck [**1**, p. 59]. We present the proof as a sequence of lemmas.

The first two of these are proved in [**7**].

LEMMA 2. *Let X_1 be a finite-dimensional Banach space and X_2 be any Banach space. Let $\{l_1, \cdots, l_n\}$ be any basis of unit vectors for X_1; if $t \in X_1 \otimes X_2$ then t has a unique representative $t = \sum_{i=1}^n l_i \otimes y_i$. Define*

$$\phi: X_1 \otimes X_2 \to \sum \oplus_n X_2 \qquad \text{by} \qquad \phi(t) = (y_1, \cdots, y_n).$$

ϕ is an algebraic isomorphism onto.

LEMMA 3. *X_1 and X_2 as in Lemma 2. If we norm $\sum \oplus_n X_2$ by defining*

$$|(y_1, \cdots, y_n)| = \sum_{i=1}^n |y_i|,$$

then ϕ as defined above is a homeomorphism of $\sum \oplus_n X_2$ and $X_1 \otimes_\gamma X_2$.

Note that since X_1 is finite-dimensional the algebraic tensor product normed with the greatest cross norm is complete.

LEMMA 4. *Suppose X_1 is a simple finite-dimensional annihilator algebra with proper involution (see [**4**]) and continuous quasi-inversion. Suppose X_2 is a Banach-$*$-algebra. Then ϕ defined in Lemma 2 is a $*$-algebra-isomorphism.*

PROOF. The assumptions on X_1 are made simply to ensure that X_1 has a basis $\{e_i\}$ consisting of orthonormal, hermitian idempotents [3, p. 330], i.e. $\{e_i\}$ satisfies

$$e_i e_j = \delta_{ij} e_j, \qquad \text{for all } i, j = 1, \cdots, n,$$

and

$$e_i^* = e_i, \qquad \text{for all } i = 1, \cdots, n.$$

If $t \in X_1 \otimes_\gamma X_2$ then as before $t = \sum e_i \otimes y_i$ and $\phi(t) = (y_1, \cdots, y_n)$. Hence $t^* = \sum e_i^* \otimes y_i^* = \sum e_i \otimes y_i^*$ and $\phi(t^*) = (y_1^*, \cdots, y_n^*) = [\phi(t)]^*$. Moreover, if $t_1 = \sum e_i \otimes x_i$ and $t_2 = \sum e_j \otimes y_j$ then

$$t_1 t_2 = \sum_{ij} e_i e_j \otimes x_i y_j = \sum_i e_i \otimes x_i y_i$$

so that

$$\begin{aligned}\phi(t_1 t_2) &= (x_1 y_1, \cdots, x_n y_n) = (x_1, \cdots, x_n)(y_1, \cdots, y_n) \\ &= \phi(t_1)\phi(t_1).\end{aligned}$$

COROLLARY 1. *X_1 and X_2 as in Lemma 4. $X_1 \otimes_\gamma X_2$ is symmetric if and only if X_2 is symmetric.*

REMARK. The assumptions about X_1 imply that X_1 is symmetric [4, p. 266].

PROOF. By Lemma 3 and Lemma 4 it suffices to show that $\sum_n \oplus X_2$ is symmetric iff X_2 is symmetric. But this is an immediate consequence of the fact that

$$\sigma(y_1, \cdots, y_n) = \bigcup_{i=1}^{n} \sigma(y_i)$$

for any $(y_1, \cdots, y_n) \in \sum \oplus_n X_2$.

We now specialize to $B^1(G, A) = L^1(G) \otimes_\gamma A$. The theory to be developed depends on the minimal ideal structure of $L^1(G)$.

If $S \subset L^1(G)$ then $(S \otimes A)_\gamma$ will denote the γ-closure of $S \otimes A$ in $L^1(G) \otimes_\gamma A$.

LEMMA 5. *If $M \subset L^1(G)$ is a minimal two-sided closed ideal, then $(M \otimes A)_\gamma$ is a closed ideal in $B^1(G, A)$, symmetric if and only if A is symmetric.*

PROOF. Since G is compact, M is a finite-dimensional simple annihilator algebra with proper involution and continuous quasi-inversion [3, VI]. Clearly $(M \otimes A)_\gamma$ is a closed ideal; if $t_1 = \sum_{i=1}^{\infty} x_i \otimes y_i \in M \otimes A$ and

$$t_2 = \sum_{j=1}^{\infty} u_j \otimes v_j \in L^1(G) \otimes_\gamma A$$

then

$$t_1 t_2 = \sum_{i,j} x_i u_j \otimes y_i v_j \in (M \otimes A)_\gamma.$$

Similarly $t_2 t_1 \in (M \otimes A)_\gamma$. It is easy to see that $M \otimes_\gamma A$ and $(M \otimes A)_\gamma$ are *-isomorphic, using the technique of Lemma 2. Since $M \otimes_\gamma A$ is symmetric iff A is symmetric, the conclusion follows by the above and Lemma 4.

LEMMA 6. *Let $\{M_i\}_{i=1}^n$ be minimal two-sided ideals of $L^1(G)$; if A is symmetric, then $\sum_{i=1}^n \oplus (M_i \otimes A)_\gamma \subset B^1(G, A)$ is symmetric.*

PROOF. If A is symmetric, then $(M_i \otimes A)_\gamma$ is symmetric (Lemma 5). The rest follows as in the proof of Corollary 1.

Now we are able to complete the proof of the theorem. Suppose A is symmetric. We must show that $B^1(G, A)$ is symmetric; from this it will follow that $B^p(G, A)$ is symmetric (Lemma 1). In accordance with [**5**] we show that hermitian elements in $B^1(G, A)$ have real spectra. Adapting a construction in [**2**, Theorem (28.53)] to the present situation we can find a net of functions $\{u_\alpha\}$ with the following properties:

(i) each u_α is complex-valued continuous, nonnegative, positive-definite and central.

(ii) $\int u_\alpha dm = 1$ for all α.

(iii) $u_\alpha * f = f * u_\alpha \to f$ for any $f \in L^1(G)$.

Clearly each u_α generates an operator T_α in $B^1(G, A)$ defined as follows

$$f = \sum x_i \otimes y_i \in B^1(G, A) = L^1(G) \otimes_\gamma A \Rightarrow$$
$$T_\alpha f = \sum u_\alpha * x_i \otimes y_i.$$

Equally clear is it that T_α approximates the identity of $B^1(G, A)$, i.e.

$$\sum u_\alpha * x_i \otimes y_i \to \sum x_i \otimes y_i.$$

Now, let f be a hermitian element of $B^1(G, A)$. We will use the notation $u_\alpha * f$ for $T_\alpha f \in B^1(G, A)$. From $\{u_\alpha * f\}$ we can pick a sequence $\{u_n * f\}$ such that

$$|u_n * f - f| < 1/n, \qquad n = 1, 2, \cdots.$$

Let $\mathfrak{F} = \{M\}$ be the collection of minimal two-sided ideals in $L^1(G)$. Since G is compact any irreducible representation of $L^1(G)$ is realizable as left translation in some $M \in \mathfrak{F}$ [3, p. 434]. Moreover, each u_n being positive definite we have that

$$u_n(\cdot) = \sum_{M \in \mathfrak{F}} c_n(M)\chi_M(\cdot)$$

where χ_M is the character of M, where $c_n(M) \geqq 0$ and where $\sum_{M\in\mathfrak{F}} c_n(M)\chi_M(e) < \infty$. Again following [2], for each u_n we can choose a finite partial sum, u_n', of the series for u_n so that $|u_n' - u_n|_\infty < 1/2n$. Setting $u_n'' = \frac{1}{2}(u_n' + u_n'^*)$ we get a continuous hermitian central trigonometric polynomial for which $|u_n'' - u_n|_\infty < 1/n$. If we normalize u_n'' to obtain v_n, i.e.

$$v_n = u_n'' / | u_n'' |_1, \qquad n = 1, \cdots,$$

then it is clear that $v_n * f \to f$. Since $v_n * f = f * v_n$ each $v_n * f$ is hermitian. Also since $v_n * f$ and f commute we can make use of [4, (1.6.17)]. Consequently, it suffices to show that $v_n * f$ has a real spectrum. But since v_n is a finite linear combination of characters it follows that

$$v_n * f \in \sum_{i \in K} \oplus \ (M_i \otimes A)_\gamma$$

where K is finite set. Lemma 6 then implies that $v_n * f$ has real spectrum. This completes the proof.

COROLLARY. *Let G be a compact group and H a locally compact group. $L^1(G\times H)$ is symmetric if and only if $L^1(H)$ is symmetric.*

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