

A CLASS OF UNIFORM CONVERGENCE STRUCTURES

G. D. RICHARDSON

ABSTRACT. In 1967, Cook and Fischer introduced in the journal *Mathematische Annalen* the notion of a uniform convergence structure, abbreviated u.c.s., for a set X . Here we consider the class Γ of u.c.s. which have the following property: a u.c.s. $I \in \Gamma$ provided there is a filter $\Phi \in I$ such that \mathfrak{F} is finer than $\Phi(x)$ for every filter \mathfrak{F} which converges to x , for each $x \in X$. Various properties of the class Γ are discussed. The main result is that a topology τ for X is regular if and only if there is an $I \in \Gamma$ such that I induces τ . Also it is shown that each $I \in \Gamma$ induces a regular topology for X .

The class Γ_0 of u.c.s. which satisfy the completion axiom was first introduced by Biesterfeldt, *Indag. Math.*, 1966. Here it is shown that $\Gamma_0 \subset \Gamma$ and a characterization of the class Γ_0 is given in terms of Cauchy filters.

1. Introduction. The reader is referred to [3] and [4] for definitions and notation used here.

Let X be any set. Denote by Γ , the class of all uniform convergence structures, abbreviated u.c.s., for X with the following property: for each $I \in \Gamma$ there is a $\Phi \in I$ such that $\tau_I(x)$ is the collection of all filters on X which are finer than $\Phi(x)$, for each $x \in X$.

Various properties of the class Γ are discussed. The main result is that any topology for X is regular if and only if there is a u.c.s. in Γ that induces the given topology.

Finally the u.c.s. for X which satisfy the completion axiom are discussed. The completion axiom was first introduced in [1].

2. A characterization for regular topologies. Let $I \in \Gamma$. Then there is a symmetric filter $\Phi \in I$ with $\Phi \leq [\Delta]$ (Φ coarser than the diagonal filter) such that $\tau_I(x)$ is the collection of all filters on X which are finer than $\Phi(x)$.

PROPOSITION 1. *Let $I \in \Gamma$. Then τ_I is a topology for X .*

PROOF. From [2], we must show that for $A(x) \in \Phi(x)$, $A = A^{-1} \in \Phi$, there is an $H \in \Phi(x)$ such that for each $y \in H$, $A(x) \in \Phi(y)$. Since $\Phi^2(x)$ converges to x , we have that $\Phi^2(x) = \Phi(x)$. Hence there is a $B \in \Phi$ such that $B^2(x) \subset A(x)$. Let $H = B(x) \in \Phi(x)$ and let $y \in H$. We claim that $A(x) \supset B(y) \in \Phi(y)$. If $z \in B(y)$, then $(y, z) \in B$. Since $y \in B(x)$, then $(x, y) \in B$ and it follows that $(x, z) \in B^2$ or $z \in B^2(x) \subset A(x)$.

Received by the editors September 9, 1969.

AMS Subject Classifications. Primary 5422, 5410; Secondary 5420.

Key Words and Phrases. Uniform convergence structures, symmetric filters, diagonal filters, ultrafilters, Cauchy filters, regular topologies.

Hence the claim follows and thus τ_I is a topology for X .

The proof of the following proposition is similar to that in [2] for uniform spaces and will be deleted here.

PROPOSITION 2. *Let $I \in \Gamma$ and $A \subset X$, $B \subset X \times X$. Then $\text{Cl}(A) = \bigcap_{V \in \Phi} V(A)$ and $\text{Cl}(B) = \bigcap_{V \in \Phi} V \circ B \circ V$.*

COROLLARY. *If $I \in \Gamma$, then $\text{Cl}(\Phi) \in I$.*

PROOF. From Proposition 2, $\text{Cl}(\Phi) \geq \Phi^3$ and hence $\text{Cl}(\Phi) \in I$.

PROPOSITION 3. *If $I \in \Gamma$, then τ_I is a regular topology.*

PROOF. We claim that $x \times \text{Cl}(\Phi(x)) \geq \text{Cl}(\Phi) \in I$ (\dot{x} is a fixed ultrafilter); of course, it suffices to show that $\{x\} \times \text{Cl}(A(x)) \subset \text{Cl}(A)$ where $A \in \Phi$. If $y \in \text{Cl}(A(x))$ and $B \in \Phi$, then $(B(x) \times B(y)) \cap A \neq \emptyset$. Hence $(x, y) \in \text{Cl}(A)$ and the claim follows. Therefore τ_I is a regular topology.

THEOREM. *Let (X, τ) be a topological space with $\eta(x)$ denoting the neighborhood filter at $x \in X$. Then τ is regular iff there is an $I \in \Gamma$ inducing τ .*

PROOF. Let $I \in \Gamma$ such that $\tau_I = \tau$. Then by Proposition 3, τ is regular.

Conversely, assume that τ is a regular topology. Denote by $\Psi_0 = \bigvee_{z \in X} (\dot{z} \times \eta(z))$, $\Phi = \Psi_0 \wedge \Psi_0^{-1}$, $F(X \times X)$ the collection of all filters on $X \times X$, and $B = \{\Phi^n \mid n = 1, 2, \dots\}$. Clearly B is a base for a u.c.s. I for X . That is, $I = \{\Psi \in F(X \times X) \mid \Psi \geq \Phi^n \text{ for some } n = 1, 2, \dots\}$ is a u.c.s. for X .

We claim that $\tau_I = \tau$. The regularity of τ implies property (3) of Theorem 1 of [5]. Hence from property (2) of the same theorem, we have that $\Phi(x) = \eta(x)$ for each $x \in X$. Thus $\dot{x} \times \eta(x) = \dot{x} \times \Phi(x) \geq \Phi$ and hence $\tau(x) \subset \tau_I(x)$.

Conversely, if $\mathfrak{F} \in \tau_I(x)$, then $\dot{x} \times \mathfrak{F} \geq \Phi^n$ for some positive integer n . Hence $\mathfrak{F} = (\dot{x} \times \mathfrak{F})(x) \geq \Phi^n(x)$. Thus we must show that $\Phi^n(x) \geq \eta(x)$. Assume $n \geq 2$. Let $N \in \eta(x)$ be open. Using the regularity of τ , there exists open neighborhoods $N_i \in \eta(x)$ ($i = 1, 2, \dots, n$) such that $x \in N_1 \subset \text{Cl}(N_1) \subset N_2 \subset \text{Cl}(N_2) \subset \dots \subset \text{Cl}(N_n) \subset N$. For each $z \in X$, define

$$\begin{aligned} N_z &= N_1 \quad \text{for } z \in N_1, \\ &= N_2 \quad \text{for } z \in \text{Cl}(N_1) - N_1, \\ &= N_{k+1} - \text{Cl}(N_{k-1}) \quad \text{for } z \in \text{Cl}(N_k) - N_k \quad (k = 2, 3, \dots, n-1), \\ &= N_{k+1} - \text{Cl}(N_k) \quad \text{for } z \in N_{k+1} - \text{Cl}(N_k) \quad (k = 1, 2, \dots, n-1), \\ &= N - \text{Cl}(N_{n-1}) \quad \text{for } z \in \text{Cl}(N_n) - N_n, \\ &= X - \text{Cl}(N_n) \quad \text{for } z \in X - \text{Cl}(N_n). \end{aligned}$$

We claim that

$$\left[\left(\bigcup_{z \in X} (\{z\} \times N_z) \right) \cup \left(\bigcup_{z \in X} (\{z\} \times N_z) \right)^{-1} \right]^n (x) \subset N.$$

Let $y \in \text{L.H.S.}$, $z_0 = x$, $z_n = y$, and A equal the set in brackets. Hence $(z_{i-1}, z_i) \in A$ for some $z_i \in X$ ($i = 1, 2, \dots, n$). By computation, one can show that $z_i \in N_{i+1}$ ($i = 1, 2, \dots, n-1$) and $y \in \text{Cl}(N_n) \subset N$. Therefore our claim follows and we have that $\Phi^n(x) = \eta(x)$ for each natural number n and each $x \in X$. Thus $\tau_I = \tau$.

PROPOSITION 4. *If τ is a compact Hausdorff topology for X , then there is exactly one $I \in \Gamma$ inducing τ .*

PROOF. From [2] we have that $I = \{\Phi \in F(X \times X) \mid \Phi \geq \mathfrak{U}\}$, where $\mathfrak{U} = \{\text{all neighborhoods of } \Delta\}$, induces τ . Of course $I \in \Gamma$. Hence if $I_1 \in \Gamma$ and induces τ , then we want to show that $I = I_1$.

Let $\Phi_1 \in I_1$ be a symmetric filter such that $\tau(x)$ is the collection of all filters on X which are finer than $\Phi_1(x)$. We claim that $\mathfrak{U} \geq \Phi_1 \circ \Phi_1$. Let $A_1 = A_1^{-1} \in \Phi_1$. Since $A_1(x) \times A_1(x) \subset A_1 \circ A_1$ for each $x \in X$, we have that $\bigcup_{x \in X} (A_1(x) \times A_1(x)) \in \mathfrak{U}$ and is contained in $A_1 \circ A_1$. Hence the claim follows and thus $I \subset I_1$.

Conversely, we claim that $\Phi_1 \geq \mathfrak{U}$. Suppose there is a $V \in \mathfrak{U}$ such that for all $A_1 \in \Phi_1$, $A_1 \cap V^c \neq \emptyset$. Assume w.l.o.g. that V is an open neighborhood of Δ . The set $\{A_1 \cap V^c \mid A_1 \in \Phi_1\}$ is a base for a filter \mathfrak{L} on $X \times X$. Since $(X \times X, \tau \times \tau)$ is compact, $(x, y) \in \text{adh}(\mathfrak{L})$ for some $x, y \in X$. Hence $(x, y) \in \text{Cl}(A_1 \cap V^c)$ for each $A_1 \in \Phi_1$. Thus $(x, y) \in \text{Cl}(V^c) = V^c$ and $(x, y) \in \text{Cl}(A_1)$ for each $A_1 \in \Phi_1$. Since $\Delta \subset V$, $x \neq y$. Also $\text{Cl}(\Phi_1)(x) = \Phi_1(x)$ and we have that $x \in \text{Cl}(\{y\})$. This contradicts τ being Hausdorff. Hence $\Phi_1 \geq \mathfrak{U}$ and thus $\Phi_1 \in I$. Let $\Psi \in I_1$. Then of course $\Psi \geq \Psi \wedge \Psi^{-1} \wedge \Phi_1 \in I_1$. By an identical argument just given for Φ_1 , we have that $\Psi \wedge \Psi^{-1} \wedge \Phi_1 \geq \mathfrak{U}$. Hence $\Psi \geq \mathfrak{U}$ and thus $I = I_1$.

3. Completion axiom. The following definition is easily seen to be equivalent to that given in [1]. A u.c.s. I is said to satisfy the completion axiom, abbreviated c.a., provided there is a base for I consisting of symmetric filters coarser than the diagonal filter such that for each Cauchy filter \mathfrak{F} on X , $\mathfrak{F} \times \mathfrak{F} \geq \Phi$ for every Φ in the base.

Let I satisfy the c.a. with base B .

PROPOSITION 5. *If I satisfies the c.a. and $\Phi \in B$, then $\tau_I(x)$ is the collection of all filters on X which are finer than $\Phi(x)$.*

PROOF. Clearly $\Phi(x) \in \tau_I(x)$. If $\mathfrak{F} \in \tau_I(x)$, then $\mathfrak{F} \wedge \dot{x} \in \tau_I(x)$. Let $A \in \Phi$. Since I satisfies the c.a., then $(F \cup \{x\}) \times (F \cup \{x\}) \subset A$ for

some $F \in \mathcal{F}$. Hence $F \subset A(x)$, which implies that $\mathcal{F} \geq \Phi(x)$ and thus the proposition follows.

Let Γ_0 denote the collection of u.c.s. for X which satisfy the c.a. From the above proposition $\Gamma_0 \subset \Gamma$. Hence each $I \in \Gamma_0$ induces on X a regular topology.

Let C_I denote the collection of all Cauchy filters on X [3].

PROPOSITION 6. *If I is any u.c.s. for X , then $I \in \Gamma_0$ iff $\bigwedge_{\mathcal{F} \in \mathcal{C}_I} (\mathcal{F} \times \mathcal{F}) \in I$.*

PROOF. Clearly the necessity follows. Conversely, if $\bigwedge_{\mathcal{F} \in \mathcal{C}_I} (\mathcal{F} \times \mathcal{F}) \in I$ then let $B = \{\Phi \in I \mid \Phi = \Phi^{-1}, \Phi \leq \bigwedge_{\mathcal{F} \in \mathcal{C}_I} (\mathcal{F} \times \mathcal{F})\}$. Since $\dot{x} \in \mathcal{C}_I$ for each $x \in X$, then $\Phi \leq [\Delta]$ for each $\Phi \in B$. Clearly B is a c.a. base for I .

I conjecture that each $I \in \Gamma_0$ induces a completely regular topology on X .

REFERENCES

1. H. J. Biesterfeldt, *Completion of a class of uniform convergence spaces*, Nederl. Akad. Wetensch. Proc. Ser. A **69** = Indag. Math. **28** (1966), 602–604. MR **34** #5052.
2. N. Bourbaki, *General topology*. Part I, Hermann, Paris and Addison-Wesley, Reading, Mass., 1966. MR **34** #5044a.
3. C. H. Cook and H. R. Fischer, *Uniform convergence structures*, Math. Ann. **173** (1967), 290–306. MR **36** #845.
4. H. R. Fischer, *Limesräume*, Math. Ann. **137** (1959), 269–303. MR **22** #225.
5. D. C. Kent, *A note on pretopologies*, Fund. Math. **62** (1968), 95–100. MR **36** #7102.

EAST CAROLINA UNIVERSITY, GREENVILLE, NORTH CAROLINA 27834