A STONE-ČECH COMPACTIFICATION FOR LIMIT SPACES

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ABSTRACT. O. Wyler [Notices Amer. Math. Soc. **15** (1968), 169. Abstract #653-306.] has given a Stone-Čech compactification for limit spaces. However, his is not necessarily an embedding.

Here, it is shown that any Hausdorff limit space (X, τ) can be embedded as a dense subspace of a compact, Hausdorff, limit space (X_1, τ_1) with the following property: any continuous function from (X, τ) into a compact, Hausdorff, regular limit space can be uniquely extended to a continuous function on (X_1, τ_1).

1. **Introduction.** The notation and definitions used here can be found in [**1**] and [**2**].

Wyler [**3**] has shown that for every limit space X, there is a continuous mapping $j: X \to BX$, where BX is regular, Hausdorff, and compact, with the following property: for every continuous mapping $f: X \to Y$, with Y regular, Hausdorff, and compact, there is a unique continuous mapping $g: BX \to Y$ such that $f = g \circ j$.

We obtain a somewhat similar result with j being an isomorphism.

2. **Compactification.** Let (X, τ) be a Hausdorff limit space and $\dot{x}$ the fixed ultrafilter containing $\{x\}$. Denote by $X_1 = \{\dot{x} \mid x \in X\} \cup \{\mathcal{V} \mid \mathcal{V}$ is a nonconvergent ultrafilter on $X\}$ and $\ddot{x}$ denotes the fixed ultrafilter on X_1 containing $\{\dot{x}\}$. Let $\mathfrak{F}$ be a filter on X and denote by $\hat{\mathfrak{F}}$ the filter on X_1 whose base is $\{\hat{F} \mid F \in \hat{F}\}$, where $\hat{F} = \{\mathcal{H} \in X_1 \mid F \in \mathcal{H}\}$.

Define $\tau_1(\dot{x}) = \{\mathcal{H} \in X_1 \mid \mathcal{H} \geqq \hat{\mathfrak{F}}$ for some $\mathfrak{F} \in \tau(x)\}$, where $x \in X$, and $\tau_1(\mathcal{V}) = \{\mathcal{H} \in X_1 \mid \mathcal{H} \geqq \hat{\mathcal{V}}\}$ for $\mathcal{V}$ a nonconvergent ultrafilter on X. From [**1**], $\hat{\mathfrak{F}} \wedge \hat{\mathcal{G}} = (\mathfrak{F} \wedge \mathcal{G})\hat{}$. Also $\ddot{x} \geqq \dot{x}\hat{}$, $\dot{\mathcal{V}} \geqq \hat{\mathcal{V}}$, and we easily see that τ_1 is a limit structure for X_1.

Our first goal is to show that (X_1, τ_1) is a compactification of (X, τ). That is, (X, τ) is embedded as a dense subspace of (X_1, τ_1).

LEMMA 1. *The space (X_1, τ_1) is Hausdorff.*

PROOF. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be any two filters on X and $\mathfrak{F}^*$ a filter on X_1 with $\mathfrak{F}^* \geqq \hat{\mathfrak{F}}_i$ $(i = 1, 2)$. It suffices to show that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are not disjoint. Let $F_i \in \mathfrak{F}_i$ $(i = 1, 2)$, then there exists an $F^* \in \mathfrak{F}^*$ such that $F^* \subset \hat{F}_1 \cap \hat{F}_2 = (F_1 \cap F_2)\hat{}$. Hence $F_1 \cap F_2 \neq \emptyset$ and thus $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are not disjoint filters.

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LEMMA 2. *The space* (X_1, τ_1) *is compact.*

PROOF. Let $\mathcal{V}^*$ be an ultrafilter on X_1. Define $\mathfrak{F}=\{F\subset X \mid \hat{F}\in\mathcal{V}^*\}$. Clearly $\mathfrak{F}$ is a filter on X. Moreover, if $A\cup B\in\mathfrak{F}$ then $(A\cup B)^\wedge = \hat{A}\cup\hat{B}\in\mathcal{V}^*$ and hence from [1] either $\hat{A}$ or $\hat{B}$ belong to $\mathcal{V}^*$. Therefore, either A or B belong to $\mathfrak{F}$ and again from [1] we have that $\mathfrak{F}$ is an ultrafilter on X.

Clearly $\mathcal{V}^*\geqq\hat{\mathfrak{F}}$. If $\mathfrak{F}\in\tau(x)$ for some $x\in X$, then we have that $\mathcal{V}^*\in\tau_1(\dot{x})$. On the other hand, if $\mathfrak{F}$ does not converge on X then $\mathfrak{F}\in X_1$ and thus $\mathcal{V}^*\in\tau_1(\mathfrak{F})$. Hence (X_1, τ_1) is compact.

Define $i: X\to X_1$ such that $i(x)=\dot{x}$. Since the restriction of the filter $\hat{\mathfrak{F}}$ to $i(X)$ is $i(\mathfrak{F})$, we have that i is an isomorphism onto $i(X)$.

Let $\mathcal{V}\in X_1$. Then $i(\mathcal{V})\geqq\hat{\mathcal{V}}$ and it follows that $i(X)$ is dense in X_1. Therefore, (X_1, τ_1) is a compactification of (X, τ).

THEOREM. *Let* $f:(X,\tau)\to(X_2,\tau_2)$ *be continuous with* (X_2,τ_2) *compact, regular, and both spaces Hausdorff. Then* (X_1, τ_1) *is a compactification of* (X,τ) *such that* f *has a unique continuous extension to* X_1.

PROOF. Only the latter part remains to be proved. Define $f_1: X_1\to X_2$ such that $f_1(\mathcal{V})=\lim f(\mathcal{V})$ for $\mathcal{V}\in X_1$. From [1], and [2], f_1 is well defined. In order to prove that f_1 is continuous, it suffices to show that for $\mathfrak{F}^*\geqq\hat{\mathfrak{F}}$, $\mathfrak{F}$ is a filter on X, then $f_1(\mathfrak{F}^*)\geqq \mathrm{cl}(f(\mathfrak{F}))$. Let $F\in\mathfrak{F}$. Then there is an $F^*\in\mathfrak{F}^*$ such that $F^*\subset\hat{F}$. We claim that $f_1(F^*)\subset\mathrm{cl}(f(F))$. If $\mathcal{V}\in F^*$, then $F\in\mathcal{V}$ and from [2], $f_1(\mathcal{V})\in\mathrm{cl}(f(F))$ and thus our claim follows. Hence f_1 is continuous.

Clearly f_1 is the unique continuous extension of f to X_1.

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