# A THEOREM ON $T$-FRACTIONS CORRESPONDING TO A RATIONAL FUNCTION

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ABSTRACT. We prove that a limitärperiodisch $T$-fraction, which corresponds to a rational function, has the property that $d_n \to -1$.

1. **Introduction.** We are going to prove that a "limitärperiodisch" $T$-fraction [1] has the property $d_n \to -1$ if it corresponds to a rational function.

This is a small contribution towards a solution of the problem raised by Perron [2]: "Notwendige und hinreichende Bedingungen dafür, dass der Thronsche Kettenbruch einer rationalen Funktion zugeordnet ist, sind nicht bekannt."

The interest in this question goes back to the classical theory: "A regular continued fraction represents a rational number if and only if it is terminating." For some types of continued fractions corresponding formally to power series, the $C$- and $P$-fractions [3], [4], an analogue result is known: They are both terminating if and only if the power series expansion is the expansion of a rational function. But contrary to these fractions, the $T$-fractions are nonterminating by definition, which seems to make the question of rationality rather complicated.

2. **$T$-fraction expansion in the holomorphic case.** Let us first describe how to expand a function into a $T$-fraction (such a description is given in [1], and the present one differs from that one in notations only).

Let $f_0$ be a complex-valued function of a complex variable, holomorphic in some region $D_0$, containing the origin, and normalized by $f_0(0)=1$, and let $\{f_n\}$ be the sequence of functions defined by

$$f_n(z) = 1 + (f_n'(0) - 1)z + z/f_{n+1}(z), \qquad z \neq 0, \tag{1}$$
$$f_{n+1}(0) = 1.$$

Then every $f_n$ will be holomorphic in some region $D_n$, containing the origin. With

$$d_n = f_n'(0) - 1, \qquad n = 0, 1, 2, \cdots, \tag{2}$$

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the continued fraction

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_n z} + \cdots \tag{3}$$

is the $T$-fraction expansion of $f_0$.

The $T$-fraction expansion is defined to be "limitärperiodisch" if $\{d_n\}$ converges.

3. **The main result.** The function $f_0$ has a purely periodic $T$-fraction expansion with period of length 1, if and only if

$$f_0(z) = 1 + d_0 z + z/f_0(z). \tag{4}$$

Here $f_0$ is rational if and only if $d_0 = -1$. (The only normalized rational function satisfying (4) is the constant function 1.)

This triviality can be extended in a nontrivial way:

THEOREM 1. *The $T$-fraction (expansion) of a normalized rational function is "limitärperiodisch" if and only if the sequence $\{d_n\}$ converges to* $-1$.

REMARK. The existence of nontrivial "limitärperiodisch" $T$-fractions of rational functions is proved in the Appendix.

To prove the theorem let us have a look at the

## 4. $T$-fraction expansion in the rational case.

PROPOSITION. *Let $f_0$ in §2 be given by the formula*

$$f_0(z) = \frac{1 + \sum_{k=1}^{m} A_k^{(0)} z^k}{1 + \sum_{k=1}^{m} B_k^{(0)} z^k} \tag{5}$$

*where $A_k^{(0)}$, $B_k^{(0)}$ are arbitrary (complex) constants, and let $\{f_n\}$ and $\{d_n\}$ be the sequences defined in* (1) *and* (2) *respectively.*

*Then, for $n = 1, 2, 3, \cdots$ we have*

$$f_n(z) = \frac{1 + \sum_{k=1}^{m} A_k^{(n)} z^k}{1 + \sum_{k=1}^{m} B_k^{(n)} z^k},$$

*where the constants $A_k^{(n)}$ and $B_k^{(n)}$ are given by the following recursion formulas*:

$$\begin{aligned} A_k^{(n)} &= B_k^{(n-1)}, \\ B_k^{(n)} &= A_{k+1}^{(n-1)} - B_{k+1}^{(n-1)} - d_{n-1} B_k^{(n-1)}, \qquad 1 \leq k < m, \\ B_m^{(n)} &= -\, d_{n-1} B_m^{n-1}, \quad d_{n-1} = A_1^{(n-1)} - B_1^{(n-1)} - 1. \end{aligned} \tag{6'}$$

PROOF. Directly from (1) and (2) by induction. ■

REMARK. In particular we have $A_k^{(n)} = B_k^{(n-1)}$, $n = 1, 2, 3, \cdots$, and from now on we thus consider "$B$"-sequences only, denoting $A_k^{(0)}$ by $B_k^{(-1)}$.

In addition, we shall find it convenient to interpret the last formula in (6′) as the second formula extended to $k=0$.

According to this we restate (6′) as

$$(6)\qquad \begin{aligned} B_k^{(n)} &= B_{k+1}^{(n-2)} - B_{k+1}^{(n-1)} - d_{n-1}B_k^{(n-1)}, \qquad 0 \leqq k < m, \\ &\qquad\qquad \text{where } B_0^{(n)} = 1 \text{ (definition)}, \\ B_m^{(n)} &= -\, d_{n-1}B_m^{(n-1)}, \\ d_{n-1} &= B_1^{(n-2)} - B_1^{(n-1)} - 1. \end{aligned}$$

5. **Proof of the theorem.** "Step of induction" given by

LEMMA 1 (LEMMA 1′). *Let $f_0$ have a representing formula* (5) *and a "limitärperiodisch" $T$-fraction* (*expansion*).

*If there is a $k \in \{1, 2, \cdots, m\}$ for which there exists an infinite subset of natural numbers, say $\mathfrak{N}_{k-1}$, such that*

$$(7)\qquad \lim_{n\to\infty;\, n\in\mathfrak{N}_{k-1}+h} \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n)}} = 1$$

*for any nonnegative integer $h$* (Lemma 1′: *for any nonpositive integer $h$*), *then there exists an infinite subset of the natural numbers, say $\mathfrak{N}_k$, such that*

$$(8)\qquad \lim_{n\to\infty;\, n\in\mathfrak{N}_k-h} \frac{B_k^{(n+1)}}{B_k^{(n)}} = 1.$$

REMARK. In expressions like (7) it is of course harmless that elements of the subsequence are undefined for a finite set of indices.

PROOF OF LEMMA 1. From (6): $B_{k-1}^{(n+1)} + d_n B_{k-1}^{(n)} = B_k^{(n-1)} - B_k^{(n)}$. Using the assumptions $\lim_{n\to\infty} d_n = d$ and (7), we obtain

$$(9)\qquad \lim_{n\to\infty;\, n\in\mathfrak{N}_{k-1}+h} \frac{B_k^{(n-1)} - B_k^{(n)}}{B_{k-1}^{(n)}} = 1 + d.$$

Two complementary situations have to be considered:

$$(\mathrm{A})\qquad B_k^{(n)}/B_{k-1}^{(n)} \to \infty \qquad \text{as } n\ (\in \mathfrak{N}_{k-1} + h) \to \infty.$$

(B) There exists an infinite subset $\mathfrak{N}'_{k-1}$ of $\mathfrak{N}_{k-1}$ such that

$$(10°)\qquad \lim_{n\to\infty;\, n\in\mathfrak{N}'_{k-1}} \frac{B_k^{(n)}}{B_{k-1}^{(n)}} = g \qquad (\in \mathbf{C}).$$

The situation (B) turns out to be the most cumbersome, and we treat this case first. A slight simplification of the method in the B-case works in the A-case.

Proof of Lemma 1 in the B-case. By limit considerations and induction we obtain a more general version of (10°):

$$(10)\qquad \lim_{n\to\infty;\, n\in\mathfrak{N}'_{k-1}+h} \frac{B_k^{(n)}}{B_{k-1}^{(n)}} = g - (1+d)h.$$

The next step is to construct a sequence $\{N_i\}$ of natural numbers, possessing the following properties:

(1) $N_i \in \mathfrak{N}'_{k-1}+i,\ N_{i+1} > N_i$,

(2) $n \in \mathfrak{N}'_{k-1}+i$ and $n \geqq N_i$ implies

$$\left| \frac{B_k^{(n-1)} - B_k^{(n)}}{B_{k-1}^{(n)}} - (1+d) \right| < \frac{1}{2^i},$$

(3) $n \in \mathfrak{N}'_{k-1}+i$ and $n \geqq N_i$ implies

$$\left| \frac{B_k^{(n)}}{B_{k-1}^{(n)}} - (g - i(1+d)) \right| < 1.$$

Such a sequence will be called a (1) (2) (3)-sequence.

The existence of (1) (2) (3)-sequence is obvious from (9) and (10) by the principle of recursive definition.

By simple verification we further have:

Let $\{N_i^0\}$ be a (1) (2) (3)-sequence, then the sequence $\{N_i^h\}_i$ defined by $N_i^h = N_{i+h}^0 - h$ is also a (1) (2) (3)-sequence.

We are now in a position to prove (8). We denote the set of elements in the (1) (2) (3)-sequence $\{N_i^0\}$ by $\mathfrak{N}'_k$, i.e.

$$\mathfrak{N}'_k = \{N_1^0, \cdots, N_{h+1}^0, \cdots, N_{h+i}^0, \cdots\}.$$

The analogue set corresponding to $\{N_i^h\}$ is $\{N_{h+1}^0 - h, \cdots, N_{h+i}^0 - h, \cdots\}$. But this is just the set $\mathfrak{N}'_k - h$ except possibly for a finite number of elements.

Then, because of the properties (2) and (3) we have

$$\lim_{n\to\infty;\, n\in\mathfrak{N}'_k-h} \frac{B_k^{(n-1)} - B_k^{(n)}}{B_{k-1}^{(n)}} = 1 + d, \qquad \frac{B_k^{(n)}}{B_{k-1}^{(n)}} \underset{n\to\infty;\, n\in\mathfrak{N}'_k-h}{\longrightarrow} \infty,$$

which implies

$$\lim_{n\to\infty;\, n\in\mathfrak{N}'_k-h} \left[\frac{B_{k-1}^{(n)}}{B_k^{(n)}} \frac{B_k^{(n-1)} - B_k^{(n)}}{B_{k-1}^{(n)}}\right] = 0$$

and

$$\lim_{n\to\infty;\, n\in\mathfrak{N}'_k-h} \frac{B_k^{(n-1)}}{B_k^{(n)}} = 1. \tag{11}$$

Putting $\mathfrak{N}_k = \mathfrak{N}'_k - 1$, we are through in the B-case. ■

PROOF OF LEMMA 1 IN THE A-CASE. The analogue of (10) is

$$\frac{B_k^{(n)}}{B_{k-1}^{(n)}} \longrightarrow \infty \qquad \text{as } n \quad (\in \mathfrak{N}_{k-1} + h) \longrightarrow \infty. \tag{12}$$

We proceed quite as in (B). The properties (1) and (2) for the sequence can be stated as in (B) omitting the primes, but (3) is modified to

$$n \in \mathfrak{N}_{k-1} + i, \qquad n \geqq N_i \ \text{ implies } \left| B_k^{(n)}/B_{k-1}^{(n)} \right| > i. \tag{$\tilde{3}$}$$

The existence of a (1) (2) ($\tilde{3}$)-sequence is obvious from (9) and (12), the heredity of the properties (1), (2) and ($\tilde{3}$) follows as before, and we can draw our conclusions just as in (B). ■

Proof of Lemma 1′ is of course analogous. Every formula is "reflected" ($h \rightsquigarrow -h$).

After this investigation our theorem is easily proved:

Of course the normalized rational function can be represented by (5) for some nonnegative integer $m$.

We assume $m > 0$ ($m = 0$ trivial). By definition $\lim_{n\to\infty} B_0^{(n+1)}/B_0^{(n)} = 1$. By use of Lemma 1, 1′ and finite induction the existence of a subsequence of $\{B_m^{(n+1)}/B_m^{(n)}\}_n$ converging to 1 is obvious. But from (6) a limit point of $\{B_m^{(n+1)}/B_m^{(n)}\}_n$ has to be equal to $-\lim d_n$. ■

6. **Final remark.** An investigation of the *convergence* of the $T$-fractions in question is outside the purpose of the present paper. It may, however, be worth mentioning, that if a rational function (5) has a "limitärperiodisch" $T$-fraction, the $T$-fraction converges to the "right" function in a certain domain. In fact the following theorem holds:

Let $f_0$ be a rational function normalized by $f_0(0)=1$ and with a "limitärperiodisch" $T$-fraction.

Take an arbitrary $\theta \in (0, 1)$, and let $D_\theta$ denote the disk $\{z; |z| \leqq \theta\}$. Remove from $D_\theta$ arbitrary neighborhoods of the poles of $f_0$ in $D_\theta$.

Then the $T$-fraction of $f_0$ converges to $f_0$ uniformly on the remaining set.

The proof of this theorem will be published later.

**Appendix.** Let $f_0$ be given by the formula

$$(13) \qquad f_0(z) = (1 + b_{-1}z)/(1 + b_0 z), \qquad b_{-1}, \quad b_0 \in \mathbf{C}.$$

In this case the formulas (6) reduce to the following ($B_1^{(n)}$ is replaced by $b_n$):

$$(14) \qquad b_{n+1} = b_n(1 + b_n - b_{n-1}), \qquad n \geqq 0,$$

$$(15) \qquad d_n = b_{n-1} - b_n - 1, \qquad n \geqq 0.$$

Observation 1. From (15) and Theorem 1 it follows:

The $T$-fraction (expansion) of (13) is "limitärperiodisch" if and only if $b_n - b_{n-1} \to 0$.

Observation 2. Let $f_0 \neq 1$ have a representation formula (13). If there exists a nonnegative integer $i$ such that $b_{i+1} = b_i$, then $b_n = 0$, $n \geqq i$.

Proof. Directly from (14) by the well-ordering principle.

Lemma 2. *Let $b_0$, $0 < |b_0| < 1$ be given. Then $b_0$ has a neighbourhood $O_\epsilon(b_0)$ such that $b_{-1} \in O_\epsilon(b_0)$ implies the convergence of the sequence $\{b_n\}$.*

Proof. Choose $K$ such that $|b_0| < K < 1$ holds and let

$$\epsilon = (1 - K) \ln(K/|b_0|).$$

We shall first prove (by induction) that the sequence $\{|b_n|\}$ is bounded by $K$: $P(n)$ $(n \geqq 0)$: $b_{-1} \in O_\epsilon(b_0)$ implies that $|b_i| < K$ for $0 \leqq i \leqq n$. $P(0)$ is trivial.

Proof that $P(n)$ implies $P(n+1)$ for all $n \geqq 0$: By (14)

$$|b_{i+1}| \leqq |b_i| (1 + |b_i - b_{i-1}|), \qquad i \geqq 0,$$

and hence

$$(16) \qquad |b_{n+1}| \leqq |b_0| \prod_{i=0}^{n} (1 + |b_i - b_{i-1}|).$$

Furthermore, by (14)

$$|b_i - b_{i-1}| = |b_{i-1}| \, |b_{i-1} - b_{i-2}|, \qquad i \geqq 1,$$

and repeating this, we have

$$(17) \qquad |b_i - b_{i-1}| = |b_0 - b_{-1}| \prod_{j=0}^{i-1} |b_j|, \qquad i \geqq 0.$$

Combining (16) and (17), we obtain

$$(18) \qquad \begin{aligned} |b_{n+1}| \leqq |b_0| (1 + |b_0 - b_{-1}|)(1 + |b_1| \, |b_0 - b_1|) \\ \cdots (1 + |b_{n-1}| \cdots |b_1| \, |b_0 - b_{-1}|). \end{aligned}$$

The induction hypothesis yields

$$\begin{aligned} |b_{n+1}| &< |b_0| \prod_{i=0}^{n-1} (1 + K^i \epsilon) < |b_0| \exp\left( \sum_{i=0}^{n-1} K^i \epsilon \right) \\ &< |b_0| \exp\left( \epsilon \sum_{i=0}^{\infty} K^i \right) = |b_0| \exp\left( \frac{\epsilon}{1 - K} \right) = K. \end{aligned}$$

Now the convergence of $\{b_n\}$ follows easily. (According to Observation 2 it is sufficient to consider the case $b_n \neq b_{n-1}$ for all $n$):

$$b_n = b_1 + (b_2 - b_1) + \cdots + (b_n - b_{n-1}).$$

Since $|(b_{n+1} - b_n)/(b_n - b_{n-1})| = |b_n| < K < 1$, the proof of lemma is established. ∎

THEOREM 2. *There exists an uncountable set of functions* (13) *with nontrivial "limitärperiodisch" T-fraction* (*expansion*).

PROOF. Choose in (13)

$$b_{-1}, b_0 \in \boldsymbol{R}, \qquad 0 < b_0 < 1, \qquad b_{-1} < b_0, \qquad b_{-1} \in O_\epsilon(b_0)$$

From Lemma 2 and Observation 1 it follows that this $f_0$ is "limitär-periodisch."

Since the sequence $\{b_n\}$ is *strictly* increasing the expansion is nontrivial. ∎

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