A CRINKLED ARC

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If X is an infinite dimensional Hilbert space, then there is a homeomorphism f of $[0, 1]$ into X such that if $0 \leqq a < b \leqq c < d \leqq 1$ then $f(b) - f(a)$ is orthogonal to $f(d) - f(c)$ i.e., each two nonoverlapping chords of the arc f are orthogonal.

In dealing with such crinkled arcs let us consistently take the point of view that an arc is a function.

The only arcs to be considered here are maps from $[0, 1]$ into Hilbert space X which are one-to-one, continuous, and have the crinkly property: each two nonoverlapping chords are orthogonal.

Let us now normalize in three additional ways. If f is a crinkled arc we may assume (by translation) that $f(0) = 0$; we may assume (by a change of scale) that $\|f(1)\| = 1$; and we may assume that $\bigvee f = X$ (by discarding $(\bigvee f)^{\perp}$). Here $\bigvee f$ is understood to be the smallest closed subspace of X containing the values of f.

All arcs to be mentioned in what follows are *normalized* in all the ways described above.

LEMMA 1. *If f is a crinkled arc, then $t \to \|f(t)\|$ is a strictly monotone continuous map of $[0, 1]$ onto $[0, 1]$; if $0 \leqq s \leqq t \leqq 1$, then $(f(s), f(t)) = \|f(s)\|^2$; and, finally, if $0 \leqq s \leqq t \leqq 1$, then $\|f(t) - f(s)\|^2 = \|f(t)\|^2 - \|f(s)\|^2$.*

PROOF. If $0 \leqq s < t \leqq 1$ then $(f(s), f(t)) = (f(s) - f(0), f(t) - f(s) + f(s)) = (f(s), f(s))$.

LEMMA 2. *The nonzero values of f (a crinkled arc) are linearly independent.*

PROOF. Suppose $0 < t_1 < t_2 < \cdots < t_n \leqq 1$ then the vectors $f(t_1)$, $f(t_2) - f(t_1), \cdots, f(t_n) - f(t_{n-1})$ are pairwise orthogonal, hence linearly independent. The vectors $f(t_1), \cdots, f(t_n)$ span the same n-dimensional space and hence they too must be linearly independent.

THEOREM 1. *Of any two normalized crinkled arcs, each is unitarily equivalent to a reparametrization of the other.*

PROOF. Suppose $f: [0, 1] \to X$ and $g: [0, 1] \to Y$ are normalized

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crinkled arcs. Given t, $0 \leqq t \leqq 1$, there is a unique t' such that $\|f(t)\| = \|g(t')\|$. This can be done because $\|g\|$ is continuous and strictly monotone, moreover, since the mapping $t \to t'$ maps $[0, 1]$ onto $[0, 1]$, it follows that every value of $\|g\|$ occurs as $\|g(t')\|$ for some t'.

Define a mapping U, first for the range of f only, by $U(f(t)) = g(t')$. Since the nonzero values of range f are linearly independent, extend U to the algebraic span of range f. The range of U will then be the algebraic span of range g, so that (since $\bigvee f = X$ and $\bigvee g = Y$) both the domain and range of U are dense in their respective spaces.

On range f, U was isometric by definition. Now if $0 \leqq s \leqq t \leqq 1$ then $\|U(f(t) - f(s))\|^2 = \|g(t') - g(s')\|^2 = \|g(t')\|^2 - \|g(s')\|^2 = \|f(t)\|^2 - \|f(s)\|^2 = \|f(t) - f(s)\|^2$, and hence U is isometric on differences such as $f(s) - f(t)$, where $s \leqq t$.

Each vector in the algebraic span of range f is a finite linear combination of orthogonal differences such as $f(t) - f(s)$, and its U image is a similar combination of $(g(t') - g(s')s)$. Since the square norm is additive for orthogonal summands, it follows that U, as defined so far, is isometric.

Extend U, by continuity, and it will become an isometry from X onto Y.

Hence we have an isometry U from X onto Y such that $U(f(t)) = g(t')$, where $t \to t'$ is a homeomorphism of $[0, 1]$ onto $[0, 1]$, a reparametrization.

References

1. Paul R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.

2. ———, *Introduction to Hilbert space*, Chelsea, New York, 1957.

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