

ON THE SUM OF TWO BOREL SETS

P. ERDÖS AND A. H. STONE¹

ABSTRACT. It is shown that the linear sum of two Borel subsets of the real line need not be Borel, even if one of them is compact and the other is G_δ . This result is extended to a fairly wide class of connected topological groups.

1. Introduction. If C and D are Borel subsets of the real line R , need $C+D$ be Borel?² Here $C+D$ denotes the set $\{x+y \mid x \in C, y \in D\}$. In the simplest cases the answer is obviously "yes"; for example if at least one of C, D is countable or open, or if both are F_σ sets. We shall show that in the next simplest case, in which C is compact and D is G_δ , the answer is "no"; $C+D$ need not be Borel.³ (It will, of course, be analytic; in fact the sum of two analytic sets is analytic, being a continuous image of their product.)

The answer to the corresponding question about the plane (with $+$ denoting vector sum) has been known for some time, though it does not appear to be in the literature. The present construction imitates the plane counterexample in the space $A \times B$, where A, B are suitable additive subgroups of R , and then transfers it to $A+B \subset R$. The axiom of choice is not required.

2. The subgroups. As was shown by von Neumann [3], if we put (1) $f(x) = \sum_{n=1}^{\infty} p(p([nx]))/p(p(n^2))$, where $p(a) = 2^a$, then the numbers $f(x), x > 0$, are algebraically independent. Clearly f is strictly increasing, and is continuous at each irrational x ; hence, if P^+ denotes the set of positive irrationals, $f(P^+)$ is homeomorphic to P^+ and therefore contains a Cantor set K .⁴ In turn, K clearly contains two (in fact, c) disjoint Cantor sets K_1, K_2 . We let A, B denote the additive subgroups of R generated by K_1, K_2 respectively. Thus

Received by the editors July 22, 1969.

AMS Subject Classifications. Primary 2810; Secondary 2635, 5401, 2210.

Key Words and Phrases. Borel set, analytic set, complete metric space, Cantor set, algebraically independent, connected topological group, absolute G_δ .

¹ The second author's contribution was supported by the National Science Foundation.

² We are indebted to Mr. B. V. Rao for calling our attention to this problem.

³ A closely related result has been obtained independently, by a different method, by C. A. Rogers [4].

⁴ By "Cantor set" we mean any space homeomorphic to the usual Cantor ternary set; that is, a compact, zero-dimensional dense-in-itself metric space. In particular, the Cantor subsets of R are just the nonempty bounded perfect nowhere dense sets.

(2) A and B are σ -compact and contain Cantor sets, and

$$A \cap B = \{0\}.$$

3. The sets.

THEOREM. *There exist a Cantor set $C \subset R$, and a G_δ subset D of R , such that $C + D$ is not Borel.*

PROOF. The subgroup A contains K_1 which contains a homeomorph P_1 of the space of irrational numbers. Take a non-Borel analytic subset E of the Cantor set K_2 (cf. [1, p. 368]). There is a continuous map g of P_1 onto E ; let G be its graph, a subset of $P_1 \times K_2 \subset A \times B$. As in [1, pp. 366, 367], G is closed in $P_1 \times B$; and P_1 is an absolute G_δ . Thus G is G_δ in $A \times B$, and therefore

(3) $(A \times B) \setminus G$ is σ -compact.

Let $F = A \times \{0\}$. Note that $F + G$ (where $+$ here refers to the group operation in the direct product $A \times B$) is not Borel in $A \times B$, because its intersection with $\{0\} \times B$ is the non-Borel set $\pi_2(G) = E$.

Now consider the homomorphism $\phi: A \times B \rightarrow R$ given by $\phi(a, b) = a + b$. Clearly ϕ is continuous and (by choice of A and B) one-to-one. We note that $\phi(F + G)$ is not Borel in R , since otherwise the continuity of ϕ would show that $\phi^{-1}(\phi(F + G))$ would be Borel in $A \times B$; but this set is $F + G$. Thus

(4) $\phi(F) + \phi(G)$ is not Borel in R .

We have, however,

(5) $\phi(F) = A = \bigcup_{m=1}^\infty A_m$ where each A_m is a Cantor set.

For we may take $A_m =$ set of all numbers of the form $a_1 + a_2 + \dots + a_m$ where $\pm a_i \in K_1$ ($i = 1, 2, \dots, m$). This is a Cantor set because it is clearly compact and perfect, and also nowhere dense (since otherwise $A = R$, contradicting (4)).

Again, $\phi(G)$ is G_δ in $A + B$, for (since ϕ is 1-1) its complement $(A + B) \setminus \phi(G)$ is the image under ϕ of $(A \times B) \setminus G$, and is therefore σ -compact, by (3). But $A + B$ is F_σ in R ; hence $\phi(G)$ is $G_{\delta\sigma}$ in R , and we may write $\phi(G) = \bigcup_{n=1}^\infty G_n$ where each G_n is a G_δ in R . Now (4) and (5) show that $\bigcup_{m,n} (A_m + G_n)$ is non-Borel; hence there exist m, n such that $A_m + G_n$ is non-Borel, and we merely take $C = A_m, D = G_n$.

4. Remarks. Mr. Rao has called to our attention that, starting from the above theorem, L. A. Rubel's method [5] will produce pathological Borel measurable functions on the real line. For instance, if $\phi(x) = \sup_{-\infty < t < \infty} |f(x+t) - f(x-t)|$, then the Borel measurability of f does not imply that of ϕ .

It may also be worth remarking that not every analytic subset of R is expressible as the sum of two (or more) Borel sets. For example, if H is an arbitrary non-Borel analytic subset of $[0, 1]$, and $L = H \cup \{3\}$, then L is not expressible in the form $X + Y$ for any non-degenerate sets X, Y . For otherwise it is easy to see that, for some $\lambda \neq 0$, $L \cap (L + \lambda)$ contains a translate of X (take $\lambda = y_1 - y_2$ where $y_1, y_2 \in Y$), and thus that $\text{diam } X < 1$. Similarly $\text{diam } Y < 1$ and so $\text{diam } (X + Y) < 2$, contradicting $X + Y = L$.

5. **More general groups.** Mycielski [2] has generalized von Neumann's construction, showing in particular that every connected topological group with a complete metric, which is either locally compact or abelian, contains an independent Cantor subset. The foregoing arguments apply virtually unchanged⁵ to show that every such group (written additively) contains two Borel sets (in fact a compact set and a G_δ) whose sum is not Borel. It would be interesting to know whether this remains true if "connected" is weakened to "nondiscrete".

REFERENCES

1. C. Kuratowski, *Topologie*. Vol. 1, 2nd ed., Monografie Mat., vol. 20, PWN, Warsaw, 1948; English transl., Academic Press, New York and PWN, Warsaw, 1966. MR 10, 389.
2. J. Mycielski, *Independent sets in topological algebras*, Fund. Math. 55 (1964), 139-147. MR 30 #3855.
3. J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. 99 (1928), 134-141.
4. C. A. Rogers, *A linear Borel set whose difference set is not a Borel set*, Bull. London Math. Soc. (to appear).
5. L. A. Rubel, *A pathological Lebesgue-measurable function*, J. London Math. Soc. 38 (1963), 1-4. MR 26 #5123.

HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, HUNGARY AND
UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627

⁵ In the nonabelian case, $A + B$ need not be a group, and ϕ need not be a homomorphism; however, we still have $\phi(F + G) = \phi(F) + \phi(G)$ because of the special nature of F .