

RINGS OF CONTINUOUS FUNCTIONS AND TOTALLY INTEGRALLY CLOSED RINGS¹

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0. Introduction. We shall prove that the ring of continuous complex-valued functions on a completely regular Hausdorff space is totally integrally closed iff the space is extremally disconnected.

X always denotes a completely regular Hausdorff space. \mathbf{R} is the field of real numbers, \mathbf{C} of complex numbers. $C(X)$ denotes the ring of all continuous \mathbf{R} -valued functions on X , $C^*(X)$ the subring of bounded functions in $C(X)$. $i = \sqrt{-1}$. Then $C(X)[i]$ (respectively, $C^*(X)[i]$) is the ring of continuous (respectively, bounded continuous) \mathbf{C} -valued functions of X .

A ring A (ring \equiv commutative ring with 1) is *totally integrally closed* if for every integral extension $h: B \rightarrow C$, the induced map $h^*: \text{Hom}(C, A) \rightarrow \text{Hom}(B, A)$ is surjective. See [1] and [4] for a detailed treatment. By [4, Theorem 1 and Proposition 5], A is totally integrally closed iff the following four conditions hold:

- (1) A is reduced,
- (2) every monic polynomial of positive degree in one indeterminate over A factors completely into monic linear factors,
- (3) distinct minimal primes of A are comaximal, and
- (4) the set of minimal primes of A , $\text{Min Spec } A$, is an extremally disconnected space in the inherited Zariski topology.

1. The result.

THEOREM. *The following conditions on a completely regular Hausdorff space X are equivalent.*

- (a) X is extremally disconnected.
- (b) $C(X)[i]$ is totally integrally closed.
- (c) $C^*(X)[i]$ is totally integrally closed.

PROOF. We show (b) \Leftrightarrow (c) \Leftrightarrow (a). Let S be the set of functions in $C^*(X)[i]$ which never vanish. It is easy to see that $C(X)[i] = S^{-1}C^*(X)[i]$, and that $C^*(X)[i]$ is integrally closed in $C(X)[i]$. By [4, Propositions 1 and 7], these two rings are totally integrally closed or not alike. Now assume (c). The obvious map $X \rightarrow \text{Spec } C^*(X)[i]$ which takes x into $\{f: f(x) = 0\}$ is an embedding, and

Received by the editors April 18, 1969.

¹ This research was partially supported by NSF grant GP-8496.

the image is dense. But $\text{Spec } C^*(X)[i]$ must be extremal [4, Proposition 3]; hence, X must be extremally disconnected.

Now assume (a). We verify conditions (1)–(4) given in the introduction for $A = C^*(X)[i]$. A is obviously reduced (no nilpotents). Condition (3) is immediate because X is extremally disconnected $\Rightarrow X$ is an F -space [2, p. 214, N4], and X is an F -space \Leftrightarrow (3) holds [2, p. 208]. (4) is immediate from [3, Theorem 5.3]. I wish to thank the referee for this short proof of (3) and (4). To prove (2), we first note that X is zero dimensional in the sense of [2, p. 246]. (A space X is zero dimensional iff, equivalently, either (1) disjoint zero sets in X have disjoint clopen neighborhoods, or (2) the Stone-Ćech compactification βX of X has a clopen basis.) X is zero dimensional iff βX is, and if X is zero dimensional, so is each cozero set $U \subset X$. For if f is a nonnegative function vanishing precisely on $X - U$, for each positive integer n , let V_n be a clopen set in X containing the zero set $f^{-1}([1/n, +\infty])$ and disjoint from the zero set $f^{-1}([0, 1/(n+1)])$: then $V_n \subset V_{n+1}$, all n , and $U = \bigcup_n (V_{n+1} - V_n)$ is a disjoint union of clopen subsets of the zero dimensional space X . We are now ready to complete the proof of the theorem by establishing

LEMMA. *If X is zero dimensional, then every monic polynomial of positive degree in one indeterminate t over $C^*(X)[i]$ splits into monic linear factors.*

PROOF. Let $F = F(t) = t^n + \sum_{j=0}^{n-1} h_j t^j \in C^*(X)[i][t]$ be monic of degree $n \geq 1$. For each x in X let $F_x(t) = t^n + \sum_{j=0}^{n-1} h_j(x) t^j \in C[t]$. We use induction on n . The case $n = 1$ is trivial. Assume that $n > 1$ and that smaller degree polynomials split (for all zero dimensional X). Let $Y = \{y \in X : \text{all roots of } F_y \text{ are equal, i.e. } F_y = (t - c_y)^n \text{ for some } c_y \in C\}$. $\sum_{j=0}^{n-2} |h_j - \binom{n}{j} (h_{n-1}/n)^{n-j}|$ vanishes precisely on Y . Hence $X - Y$ is a cozero set and is zero dimensional. Suppose $F|_{X - Y}$ splits. Then we can extend each root function continuously (uniquely) to X by giving it the value c_y for each $y \in Y$. Continuity here follows from the Hurwitz theorem [5, p. 4, Theorem (1.5)]. Hence, we may assume without loss of generality that $Y = \emptyset$.

By considering $F(t - (1/n)h_{n-1})$ instead of F we may also assume that $h_{n-1} = 0$. Let $h = \sum_{j=0}^{n-2} |h_j|^{1/n-j} \in C^*(X)$. Since $Y = \emptyset$, the roots of F_x are never all zero, and hence h never vanishes. Let $G = t^n + \sum_{j=0}^{n-1} g_j t^j = h^{-n} F(ht)$, so that $g_j = h_j/h^{n-j}$ (in particular, $g_{n-1} = 0$). An easy computation shows $|g_j| \leq 1$ for each j , so that $G \in C^*(X)[i][t]$, and it is clear that if G splits, so does F . Since $\sum_{j=0}^{n-2} |g_j|^{1/n-j} = 1$, we may also assume without loss of generality that $h = 1$.

We use a superscript e to denote the unique continuous extension of an element of $C^*(X)[i]$ to βX , and also to denote the result of extending all the coefficients of a polynomial. Clearly, it suffices to show that $F^e \in C^*(\beta X)[i][t]$ splits. We still have $h_{n-1}^e = 0$. Moreover, since $h = 1$, $H^e = \sum_{j=0}^{n-2} |h_j^e|^{1/n-j} = 1$, so that the coefficients of F^e never vanish simultaneously. Since $h_{n-1}^e = 0$, the roots of F_x^e are never all equal, for then they would all have to be zero. Thus, the hypothesis of $Y = \emptyset$ is preserved if we pass to consideration of βX instead of X .

Thus, we may assume without loss of generality that X is compact and that F_x does not have n equal roots at any point. It now suffices to show that each point $x \in X$ has some neighborhood U such that $F|_U$ splits over U , for we then get a finite cover of X by clopen sets on which F splits. Let $x \in X$ be arbitrary. Partition the roots of F_x into two disjoint nonempty sets A, B , and choose $\epsilon > 0$ less than half the distance of A from B . For some sufficiently small open neighborhood U of x , for each $u \in U$ there will be a corresponding partition $A(u), B(u)$ of the roots of F_u , where, counting multiplicities, $A(u)$ has the same number of elements as A and $B(u)$ as B . This follows from the Hurwitz theorem: $A(u)$ consists of those roots within distance ϵ of some element of A , and similarly for $B(u)$. Let

$$G_u = \prod_{a \in A(u)} (t - a)^{m(a)}$$

and

$$H_u = \prod_{b \in B(u)} (t - b)^{m(b)}$$

where $m(c)$ is the multiplicity of c as a root of F_u . It follows easily from the Hurwitz theorem that the coefficients of G_u and H_u depend continuously on u , so that F factors nontrivially $F = GH$ over U . By the induction hypothesis, G and H both split over U .

Note that the conclusion of the lemma fails if we let $X = \mathbf{C}$ or $X = \{c \in \mathbf{C} : |c| = 1\}$. Let h be the inclusion map of X into \mathbf{C} , and let $F(t) = t^n - h$ for any integer $n \geq 2$. The referee has pointed out the following startling example where X has a basis of clopen sets but the lemma fails: In the construction of the space Δ_1 in 16M on p. 264 of [2], use the unit circle of the complex plane instead of $[0, 1]$, and let h be the restricted product projection from the resulting space X to the circle. Then once more $F(t) = t^n - h$, $n \geq 2$, does not split.

ADDED IN PROOF. The lemma follows from the results of R. S. Countryman, *On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed*, Pacific J. Math. **20** (1967), 433–448.

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