

SMALL REGULAR LOCAL NOETHER LATTICES. I¹

KENNETH P. BOGART

ABSTRACT. In a recent paper the author has discussed the structure of regular local Noether lattices. In this paper it is proved that if a regular local Noether lattice has precisely 3 minimal primes, then it is isomorphic to RL_3 , the lattice of the lattice of ideals of $F[x_1, x_2, x_3]$ generated by the principal ideals (x_1) , (x_2) , and (x_3) under join and multiplication.

1. Introduction. Regular local Noether lattices were introduced and studied in some detail in [1]. That paper introduced a collection $\{RL_n\}$ of structurally simple regular local Noether lattices and described the relationship between the structure of the lattices RL_n and the structure of regular local Noether lattices in general. The main result of this paper indicates that this relationship may be much stronger than expected. We show that if a regular local Noether lattice has precisely three minimal primes, then it must be RL_3 .

Besides the fact that it is a surprise in light of the large variety of regular local rings, this result is significant for two reasons. First, Professor Dilworth has asked whether there are significant classes of Noether lattices (defined in lattice theoretic terms) which can be imbedded in rings in such a way that their essential structure is inherited from the lattices of ideals of the rings. In [2] the author showed that this is not the case for local Noether lattices (even local lattices with a very simple multiplication). The relation between regular local Noether lattices and polynomial rings described in [1] indicates that such an imbedding might exist for regular local Noether lattices. The result given here supports that conjecture. This result also proves the smallest nontrivial case of the general conjecture that if a regular local Noether lattice has only finitely many minimal primes, then it is one of the lattices RL_n . The proof of this case uses the unique factorization theorem for principal elements of a 2-dimensional regular local Noether lattice; thus the general case is probably related to a general unique factorization theorem (which has not been proved yet) for regular local Noether lattices.

Presented to the Society, October 25, 1969; received by the editors July 21, 1969.

AMS Subject Classification. Primary 0670; Secondary 0630, 1320, 1393, 1395.

Key Words and Phrases. Noether lattice, local Noether lattice, regular local Noether lattice, Noether lattice imbedding, unique factorization.

¹ This research was supported in part by the National Science Foundation under grant number GP 12291.

The notation and terminology in this paper will be the same as that of [1]; in particular, E, F, H, K and N will denote principal elements.

2. Main result.

THEOREM. *Let L be a regular local Noether lattice with precisely 3 minimal primes. Then $L = RL_3$.*

PROOF. L cannot have dimension 1, for any regular local Noether lattice of dimension 1 is isomorphic to RL_1 [1], and thus has precisely one minimal prime. If L has dimension 3, then L has 3 regular parameters, and since these are minimal primes, they are the only minimal primes. Since every element of L must be a join of products of powers of these primes, it is obvious that the imbedding of RL_3 in L is an isomorphism of RL_3 onto L .

Suppose L has dimension 2. By [1] there is a Noether-lattice imbedding of RL_2 in L . Thus L has principal elements E and F such that the unique maximal element M of L may be represented as $M = E \vee F$. Also, E and F are minimal primes of L .

It is known that minimal primes are principal in regular local Noether lattices of dimension 2; an elementary proof of this fact (which is equivalent to the unique factorization theorem for principal elements) is the following. Let P be a minimal prime of L . Since L/E is isomorphic to RL_1 [1], there is, by the Kuroš-Ore theorem, a principal element $H \leq P$ such that

$$E \vee P = E \vee H.$$

Then

$$P = P \wedge (E \vee H) = H \vee (P \wedge E) = H \vee (P:E)E = H \vee PE$$

by modularity and the fact that P is prime. If N is any principal element contained in P , then $P \geq H \vee NE$ which means that

$$I = P:N = (H \vee NE):N = H:N \vee E.$$

Since L is local and $E < M$, $H:N = I$ which means that $N \leq H$ and $H = P$. Thus in our case L has E, F and H as its 3 minimal primes.

The proof of the theorem from now on relies heavily on the arithmetic that can be developed with E, F , and H using the operations of meet, join, and multiplication. Since L/E and L/F are isomorphic to RL_1 [1],

$$(2.1) \quad E \vee H = E \vee F^p \quad \text{for some positive integer } p \text{ and}$$

$$(2.2) \quad F \vee H = F \vee E^r \quad \text{for some positive integer } r.$$

The modularity of L is the most crucial tool used from now on. First, we prove that

$$(2.3) \quad E^r \vee F^p = E^r \vee H = F^p \vee H.$$

The remainder of the proof consists of using these identifications and the modular law to obtain a contradiction.

To prove (2.3) we use the fact that comparable complements of an element in a modular lattice are equal. First, note that each element of L is a join of products of powers of E , F , and H , since the primaries associated with these primes are powers of them and the meet of relatively prime principal elements is their product.

Now to prove the first equation in (2.3) we assume inductively that

$$(2.4) \quad E^s \vee F^p = E^s \vee H \quad \text{for some } s < r$$

and prove that 2.4 is true with s replaced with $s+1$. Note that (2.4) becomes (2.1) when $s=1$. If $E^{s+1} \vee F^p \neq E^{s+1} \vee H$, then they are not comparable. To see this, note that

$$E^s \vee (E^{s+1} \vee F^p) = E^s \vee (E^{s+1} \vee H)$$

and

$$\begin{aligned} E^s \wedge (E^{s+1} \vee F^p) &= E^{s+1} \vee (E^s \wedge F^p) = E^{s+1} \vee E^s F^p \\ &= E^s (E \vee F^p) = E^s (E \vee H) = E^{s+1} \vee E^s H \\ &= E^{s+1} \vee (E^s \wedge H) = E^s \wedge (E^{s+1} \vee H); \end{aligned}$$

since E^s has unique comparable complements in the lattice $(E^s \vee H)/(E^{s+1} \vee E^s H)$, the elements $E^{s+1} \vee F^p$ and $E^{s+1} \vee H$ cannot be comparable. Thus either

$$(2.5) \quad E^{s+1} \vee F^p = E^{s+1} \vee H$$

or

$$(2.6) \quad E^{s+1} \vee F^p \vee H > E^{s+1} \vee H.$$

We shall show that the elements on the left- and right-hand side of (2.6) are complements of E^s in $(E^s \vee H)/(E^{s+1} \vee E^s H)$, so that (2.5) must hold. First,

$$E^s \vee (E^{s+1} \vee F^p \vee H) = E^s \vee H = E^s \vee (E^{s+1} \vee H).$$

Also,

$$\begin{aligned} E^s \wedge (E^{s+1} \vee F^p \vee H) &= E^{s+1} \vee [E^s \wedge (F^p \vee H)] \\ &= E^{s+1} \vee E^s X \end{aligned}$$

where $X = (F^p \vee H) : E^s$. If $X \leq E \vee H$,

$$(2.7) \quad \begin{aligned} E^s \wedge (E^{s+1} \vee F^p \vee H) &\leq E^{s+1} \vee E^s(E \vee H) \\ &= E^{s+1} \vee E^s H = E^s \wedge (E^{s+1} \vee H). \end{aligned}$$

Now let N be a principal element such that $N \leq X$, but $N \not\leq E \vee H$. Then $N = F^t$, for N cannot have a factor which is a power of E or H . Also, $t < p$, for $F^p \leq E \vee H$. Thus

$$E^s F^t \leq E^s X = E^s \wedge (F^p \vee H)$$

which implies that

$$I = (F^p \vee H) : E^s F^t = (F^{p-t} \vee H) : E^s$$

which implies that $E^s \leq H \vee F^{p-t}$. Thus $F \vee H \geq E^s$, so that

$$I = (F \vee H) : E^s = (F \vee E^r) : E^s = F \vee E^{r-s},$$

which is impossible. Thus for each principal element $N \leq X$, we have $N \leq E \vee H$. This means that (2.7) must hold. Thus (2.6) cannot hold since comparable complements are equal in a modular lattice. This proves the first equation of (2.3). Similarly, $E^r \vee F^p = F^p \vee H$, so that (2.3) holds.

The next part of the proof uses the equations

$$(2.8) \quad (E^{3r} \vee F^{3p})(E^r \vee F^p)^2 = (E^r \vee F^p)^5 = (E^r \vee F^p)^2(E^r \vee H)^3.$$

The first equation of (2.8) is proved by simply multiplying out the left-hand side of (2.8). The second equation results from equations (2.3).

By (2.8), $(E^r \vee F^p)^5 \geq F^{2p} H^3$ so that

$$E^{5r} \vee H^3 F^{2p} = (E^{5r} \vee H^3 F^{2p}) \wedge (E^r \vee F^p)^5.$$

Applying 2.3 again, we obtain

$$E^{5r} \vee H^3 F^{2p} = (E^{5r} \vee H^3 F^{2p}) \wedge (E^{3r} \vee F^{3p})(E^r \vee H)^2.$$

Multiplying the right-hand side out and using the modular law,

$$E^{5r} \vee H^3 F^{2p} = E^{5r} \vee [(E^{5r} \vee H^3 F^{2p}) \wedge A],$$

with $A = E^{4r} H \vee E^{3r} H^2 \vee E^{2r} F^{3p} \vee E^r F^{3p} H \vee F^{3p} H^2$. By the Kuroš-Ore theorem, there is a principal element $K \leq A$ such that

$$(2.9) \quad E^{5r} \vee H^3 F^{2p} = E^{5r} \vee K.$$

However, we shall show that $K \neq H^3 F^{2p}$ by showing that $H^3 F^{2p} \not\leq A$. To see this, observe that

$$\begin{aligned}
 A : H^3 F^{2p} &= (E^{4r} \vee E^{3r} H \vee E^{2r} F^{3p} \vee E^r F^{3p} \vee F^{3p} H) : H^2 F^{2p} \\
 &= (E^{4r} \vee E^{3r} H \vee E^r F^{3p} \vee F^{3p} H) : H^2 F^{2p} \\
 &= [E^{3r} (E^r \vee H) \vee E^r F^{3p} \vee F^{3p} H] : H^2 F^{2p} \\
 &= [E^{3r} (E^r \vee F^p) \vee E^r F^{3p} \vee F^{3p} H] : H^2 F^{2p} \\
 &= (E^{4r} \vee E^{3r} \vee E^r F^{2p} \vee H F^{2p}) : H^2 F^p \\
 &= (E^{3r} \vee E^r F^p \vee H F^p) : H^2 \\
 &= [E^r (E^{2r} \vee F^p) \vee H F^p] : H^2 \\
 &= \{ E^r [(E^r \vee F^p)^2 \vee F^p] \vee H F^p \} : H^2 \\
 &= \{ E^r [(H \vee F^p)^2 \vee F^p] \vee H F^p \} : H^2 \\
 &= (E^r H^2 \vee E^r F^p \vee H F^p) : H^2 \\
 &= (E^r H \vee E^r F^p \vee F^p) : H \\
 &= E^r \vee F^p \neq I.
 \end{aligned}$$

Thus $H^3 F^{2p} \not\leq A$.

Now, since $E \vee H^3 F^{2p} = E \vee K$, we must have $K = F^i H^j$ since $H^3 F^{2p} \not\leq E$ and therefore $K \not\leq E$, so K has only factors of F and H . Write $i = kp + n$, with $n < p$. We have

$$E \vee H^5 = E \vee H^3 (E \vee H)^2 = E \vee H^3 F^{2p} = E \vee F^{kp} H^j F^n,$$

so that $j \leq 5$, since if $j > 5$, residuation by H^5 leads to a contradiction. Thus

$$E \vee H^{5-j} = E \vee F^{kp} F^n.$$

But $E \vee H^{5-j} = E \vee (E \vee H)^{5-j} = E \vee F^{p(5-j)}$. Thus $n = 0$, for the elements $E \vee F^n$ are different for different n . Then K is one of the elements $E^{5p}, H F^{4p}, H^2 F^{3p}, H^4 F^p, H^5$. Then by substitution and residuation in equation (2.9), we obtain one of the 3 equations:

$$(2.10) \quad E^{5r} \vee H^3 = E^{5r} \vee F^{3p}$$

$$(2.11) \quad E^{5r} \vee H^2 = E^{5r} \vee F^{2p}$$

or

$$(2.12) \quad E^{5r} \vee H = E^{5r} \vee F.$$

All of these are impossible. For example, if (2.10) holds then,

$$E^{5r} \vee F^{3p} \geq H^3 \vee F^{3p} \geq (H \vee F^p)^5 \geq (E^r \vee F^p)^5 \geq E^{4r} F^p$$

which implies that

$$I = (E^{5r} \vee F^{3p}) : E^{4r} F^p = E^r \vee F^{2p}$$

which is impossible. Similar arguments show that (2.11) and (2.12) do not hold. This means that L cannot have dimension 2, so that it has dimension 3 and is isomorphic to RL_3 . This proves the theorem.

The computations used in proving the theorem are quiet tedious and seem unmotivated. The motivation will appear if the reader tries to form a sublattice of the lattice of ideals of $F[x, y]$ (F a field) using only products of powers of (x) , (y) and $(x+y)$ in the way RL_n was formed from $F[x_1, \dots, x_n]$. It seems reasonable to hope that a less tedious proof of the theorem would provide a great deal of insight into the structure of regular local Noether lattices.

REFERENCES

1. K. P. Bogart, *Structure theorems for regular local Noether lattices*, Michigan Math. J. **15** (1968), 167–176. MR **37** #2642.
2. ———, *Noether lattices with trivial multiplications*, Proc. Amer. Math. Soc. (to appear).

DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755