THREE SETS OF CONDITIONS ON RINGS[1]

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ABSTRACT. We define a set of conditions $\mathfrak{L}_m$ on a ring R using the notion of R-dependence of elements. We prove that $\mathfrak{L}_1, \mathfrak{L}_2, \cdots$ is a strictly decreasing sequence of conditions. Two other sequences of conditions are considered and we prove that they are also strictly decreasing and we obtain their relation to $\mathfrak{L}_m$.

1. **Introduction.** In this paper we deal with the following set of conditions on a nonzero ring R.

$\mathfrak{L}_m$: *For each* $d=1, \cdots, m$ *between any* d *elements in* R *which are left* R-*dependent there exists one which depends on the others.*

We shall identify a condition with the class of rings in which it holds and by this convection it is clear that $\mathfrak{L}_{m-1} \supseteq \mathfrak{L}_m$. The aim of this paper is to construct for each m a ring R satisfying $\mathfrak{L}_{m-1}$ but not satisfying $\mathfrak{L}_m$ and thus obtaining that the sequence $\mathfrak{L}_1, \mathfrak{L}_2, \cdots$ is strictly decreasing.

We consider also two other sets of conditions on rings. One of them N_m: If a matrix $C \in R_m$ is nilpotent then $C^m = 0$. The other is the class of m-firs [**1**, p. 10] which we denote by $\mathfrak{F}_m$. We shall prove that the sequences $\{\mathfrak{N}_m\}$ and $\{\mathfrak{F}_m\}$ are also strictly decreasing.

First we shall obtain some properties of $\mathfrak{L}_m$. Then we shall show that $\mathfrak{L}_m \subseteq \mathfrak{F}_m$. By a result of [**5**] which is based on results of [**2**] it will follow that $\mathfrak{F}_m \subseteq \mathfrak{N}_m$ and hence $\mathfrak{L}_m \subseteq \mathfrak{N}_m$. From the definition of $\mathfrak{F}_m$ it follows that $\mathfrak{F}_{m-1} \supseteq \mathfrak{F}_m$. Also by [**5**, Lemma 9] we have $\mathfrak{N}_{m-1} \supseteq \mathfrak{N}_m$. Now our main result is obtained by proving that the rings $\mathcal{R}$ constructed in [**4**] with a fixed $m \geqq 2$ and an arbitrary $k \geqq m$ satisfy $\mathfrak{L}_{m-1}$. Since by [**4**, Theorem 1] $\mathcal{R} \in \mathfrak{N}_m$ it follows at once that all the three sequences $\{\mathfrak{L}_m\}$, $\{\mathfrak{F}_m\}$ and $\{\mathfrak{N}_m\}$ are strictly decreasing. The results for $\{\mathfrak{F}_m\}$ and $\{\mathfrak{N}_m\}$ may be deduced also from a result of Bergman [**1**]. He has proved without giving the details that the ring R constructed in [**4**] with $k \geqq m \geqq 2$ satisfies $m-1$-term algorithm [**1**, p. 25] and it is therefore a $m-1$-fir. Since $\mathfrak{F}_{m-1} \subseteq \mathfrak{N}_{m-1}$ it follows that $R \in \mathfrak{N}_{m-1}$. Bergman's result is based on general theorems but our proof is direct and implies also the results for $\{\mathfrak{L}_m\}$.

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2. **The conditions** $\mathfrak{L}_m$. We begin with some properties of rings which satisfy $\mathfrak{L}_m$. We have noted above that $\mathfrak{L}_1 \supseteq \mathfrak{L}_2 \supseteq \mathfrak{L}_3 \supseteq \cdots$. Clearly $\mathfrak{L}_1$ is the class of integral domains. If $R \in \mathfrak{L}_m$, $m \geqq 2$, then $R - \{0\}$ is a semigroup with an identity and is embeddable in a group. This follows since by definition $R \neq \{0\}$ and $R - \{0\}$ satisfies Doss' condition [**3**]. Indeed, if $ra = sb \neq 0$, $r, a, s, b \in R$, then a and b are left R-dependent and hence one of them depends on the other which means that it is a multiple of the other. In particular if $0 \neq a \in R$ then since $aa = aa$ it follows that $a = ea$ for some $e \in R$ and since R is an integral domain e is an identity which will be denoted by 1.

Now we prove a lemma which shows that $\mathfrak{L}_m$ is equivalent to its right dual condition (the case $n = 1$) and hence it follows that the left dual assertion of the lemma also holds. This lemma will be used to prove directly that $\mathfrak{L}_m \subseteq \mathfrak{N}_m$.

LEMMA 1. *If $R \in \mathfrak{L}_m$, then for each $n \geqq 1$ and for each $d \leqq m$, if $v_1, \cdots, v_d \in R^{(n)}$ are right R-dependent then one of them depends on the others.*

PROOF. Since for $d \leqq m$ we have $\mathfrak{L}_d \supseteq \mathfrak{L}_m$ it is clear that it suffices to prove the assertion for $d = m$. We shall do this by induction on m. Let $v_i = (a_{i1}, \cdots, a_{in})$, $i = 1, \cdots, m$ and $\sum_{i=1}^m v_i b_i = 0$ for some $b_1, \cdots, b_m \in R$ not all of them zero. If $m = 1$ then since R is an integral domain and $b_1 \neq 0$ it follows $v_1 = 0$. Let $m \geqq 2$ and assume the result holds for $m - 1$. If some $v_i = 0$ then it depends on the others. Hence we may assume $a_{1j} \neq 0$ for some j. But $\sum_{i=1}^m a_{ij} b_i = 0$ shows that $b_1, \cdots, b_m$ are left R-dependent and since $R \in \mathfrak{L}_m$ one of them, say b_m, depends on the others. Hence $b_m = \sum_{i=1}^{m-1} c_i b_i$ for some $c_1, \cdots, c_{m-1} \in R$ and not all of the $b_1, \cdots, b_{m-1}$ are zero since otherwise $b_m = 0$ and we have assumed that at least one of the b_i's is $\neq 0$. This implies

$$\sum_{i=1}^{m-1} (v_i + v_m c_i) b_i = \sum_{i=1}^{m-1} v_i b_i + v_m \sum_{i=1}^{m-1} c_i b_i = \sum_{i=1}^{m} v_i b_i = 0$$

which shows that $v_i + v_m c_i$, $i = 1, \cdots, m - 1$ are right R-dependent and by the induction hypothesis it follows that one of them, say $v_1 + v_m c_1$ depends on the others and hence v_1 depends on $v_2, \cdots, v_m$.

In the proof of the following lemma we use the previous one although it is possible to obtain the result directly. Note that $\mathfrak{F}_1$ is the class of integral domains with identity and $\mathfrak{L}_1$ is the class of all integral domains.

LEMMA 2. *For each $m \geqq 2$, $\mathfrak{L}_m \subseteq \mathfrak{F}_m$.*

PROOF. Let $R \in \mathfrak{L}_m$ and $m \geqq 2$ then $1 \in R$. By [**1**, p. 8] it suffices to show that if $a_1, \cdots, a_m \in R$ are right R-dependent then there exists an invertible matrix $U \in R_m$ such that at least one component of $(a_1, \cdots, a_m)U$ is zero. By the previous lemma one of the a_i's, say a_1, depends on the others. Hence $a_1 = \sum_{j=2}^m a_j c_j$ for some $c_2, \cdots, c_m \in R$. The matrix $U = I - \sum_{j=2}^m c_j E_{j1}$ is invertible in R_m. Indeed, it is easily verified that $I + \sum_{j=2}^m c_j E_{j1} = U^{-1}$. Now the result follows since the first component of $(a_1, \cdots, a_m)U$ is $a_1 - \sum_{j=2}^m a_j c_j = 0$.

The proof of Theorem 14 in [5] shows in fact that if R is a m-fir then R satisfies the condition $\mathfrak{N}_m$. Hence we have

THEOREM 3. *For each* $m \geqq 2$, $\mathfrak{L}_m \subseteq \mathfrak{F}_m \subseteq \mathfrak{N}_m$.

Now we prove directly that $\mathfrak{L}_m \subseteq \mathfrak{N}_m$. Let us call a matrix nil triangular if it is triangular with zero diagonal elements. If $R \in \mathfrak{L}_m$, $m \geqq 2$, then each nilpotent matrix of R_m is similar to a nil triangular one. (For $m = 1$ the result is trivial.) Indeed, if $0 \neq A \in R_m$ and $A^k = 0$ but $A^{k-1} \neq 0$ it follows from $AA^{k-1} = 0$ that the m column vectors of A are right R-dependent, hence by Lemma 1, one of them, say the first, depends on the others. From this one constructs as in the proof of Lemma 2 an invertible matrix U such that the first column of $U^{-1}AU$ is 0. The lower left corner of order $m-1$ in $U^{-1}AU$ is also nilpotent and the result follows by induction. Now if $R \in \mathfrak{L}_m$, $m \geqq 2$ and $A \in R_m$ is nilpotent, it is similar to a nil triangular matrix and since the mth power of a nil triangular matrix is 0, it follows that $A^m = 0$, hence $R \in \mathfrak{N}_m$.

3. **A ring which satisfies** $\mathfrak{L}_{m-1}$. In [4] we have constructed a ring $\mathfrak{R}$ in the following way: Let $F[[x_1, \cdots, x_m]] = F[[x]]$ be the ring of formal series with m indeterminates over the field $F = \{0, 1\}$. (The same result may be obtained if F is an arbitrary field.) $\mathfrak{P}$—the subring of those series all of whose homogeneous components of odd degree are zero. $\mathfrak{J}$—the ideal in $\mathfrak{P}$ generated by the elements $x_i y^k x_j$, $1 \leqq i$, $j \leqq m$, $y = x_1^2 + \cdots + x_m^2$, and k a fixed integer $\geqq m$. The ring $\mathfrak{R}$ is the quotient ring $\mathfrak{P}/\mathfrak{J}$. We shall prove

THEOREM 4. *The ring* $\mathfrak{R}$ *satisfies the condition* $\mathfrak{L}_{m-1}$.

We have to show that if d ($\leqq m-1$) elements in $\mathfrak{R}$ are $\mathfrak{R}$-left dependent, then one of them depends on the others. In what follows we shall write $\equiv$ meaning $\equiv$ mod $\mathfrak{J}$. Since $\mathfrak{R} = \mathfrak{P}/\mathfrak{J}$ we may consider elements in $\mathfrak{P}$ which are dependent over $\mathfrak{P}$ mod $\mathfrak{J}$. Thus, let $g_1, \cdots, g_d \in \mathfrak{P}$ and $f_i, \cdots, f_d \in \mathfrak{P}$ not all of them $\equiv 0$ such that $\sum_{i=1}^d f_i g_i \equiv 0$. Without loss of generality we may assume that $f_1, \cdots, f_d$ are special series in the sense of [4, p. 121]. Indeed, if this is not the case we may interchange f_1 by $S(f_i)$ = the unique special element of $f_i + \mathfrak{J}$. Recall that in [4] we have defined the value of a special element $0 \neq p$

$=\sum p^{(\alpha)}$ by $v(p)=\gamma$ if $p^{(\gamma)}$ is the homogeneous component of lowest degree which is $\neq 0$ and $v(0)=\infty$. Now, since $f_1, \cdots, f_d$ are special and not all of them 0, it follows that $\min\{v(f_i)\mid i=1, \cdots, d\}$ is finite and assume it equals $v(f_1)$. If $v(f_1)=0$, then $f_1=1+f$ such that $v(f)>0$ and hence f_1 is invertible and its inverse $f_1^{-1}=1+f+f^2+\cdots$ belongs to $\mathcal{P}$. Multiplying $\sum_{i=1}^{d} f_i g_i \equiv 0$ by f_1^{-1}, we obtain that g_1 depends on $g_2, \cdots, g_d$ mod $\mathfrak{J}$.

Let $v(f_1)=2\beta>0$ and we shall prove that there exist $t_1, \cdots, t_d \in \mathcal{P}$ with $t_1=1$ such that $\sum_{i=1}^{d} t_i g_i \equiv 0$ which is the desired result. We obtain this in four steps.

(A) *If for some* $h\geqq 0$ *and* $v\in F[[x]]$ *we have*

$$\sum_{i=1}^{d} f_i g_i + x_j y^h v \equiv 0 \tag{1}$$

then there exist $f_1', \cdots, f_d'$ *all of whose monomials begin with* x_1 *and* $2\beta=v(f_1')=v(f_i')$ *and* $v'\in F[[x]]$ *such that*

$$\sum_{i=1}^{d} f_i' g_i + x_1 y^h v' \equiv 0.$$

PROOF. Since $v(f_i)\geqq v(f_1)>0$ we may write $f_i=\sum_{j=1}^{m} x_j f_{ij}$. Clearly $v(f_i)=\min\{v(x_j f_{ij})\mid j=1, \cdots, m\}$. Let j_0 be such that $v(f_1)=v(x_{j_0} f_{1j_0})$ and the result follows if we take $f_i'=x_1 f_{ij_0}$. Indeed $v(f_i')\geqq v(f_i)\geqq v(f_1)=v(f_1')$ and as in [**4**, Lemma 16] we obtain

$$\sum_{i=1}^{d} x_{j_0} f_{ij_0} g_i + x_{j_0} y^h v' \equiv 0$$

with $v'=v$ if $j_0=j$ and $v'=0$ if $j_0\neq j$. Hence we have also

$$\sum_{i=1}^{d} f_i' g_i + x_1 y^h v' = \sum_{i=1}^{d} x_1 f_{ij_0} g_i + x_1 y^h v' \equiv 0.$$

(B) *If in* (1) *all the monomials of* f_i *begin with* x_1, $x_j=x_1$, *and* $0<h\leqq k$, *then for* $j=1, \cdots, m$ *there exist* $w\in F[[x]]$ *and* f_{ij}, $i=1, \cdots, d$, *with* $v(f_i)-2\leqq v(f_{ij})$ *and for some* j_0,$v(f_1)-2=v(f_{1j_0})$ *such that*

$$\sum_{i=1}^{d} f_{ij} g_i + x_j y^{h-1} w \equiv 0, \qquad j=1, \cdots, m. \tag{2}$$

PROOF. Since the monomials of (1) begin with x_1 we obtain, by the analog of [**1**, Lemma 13] for series, that there exists $u\in F[[x]]$ such that $x_1 y^k u \in \mathfrak{J}$ and the monomials of $\sum_{i=1}^{d} f_i g_i + x_1 y^h v + x_1 y^k u$ do not begin with $x_1 x_m^{2k}$. Let $w=v+y^{k-h}u$ and write $f_i=\sum_{j=1}^{m} x_1 x_j f_{ij}$. By

[**4**, Lemma 3(c)] for series $x_1x_jf_{ij}$ is special. Clearly $v(f_i) = \min\{v(x_1x_jf_{ij})\,|\,j=1,\cdots m\}$, hence $v(f_i) \leqq v(f_{ij})+2$. Let j_0 be such that $v(f_1)-2=v(f_{1j_0})$. We have

$$0 \equiv \left(\sum_{i=1}^{d} f_i g_i + x_1 y^h v\right) + x_1 y^k u = \sum_{i=1}^{d} f_i g_i + x_1 y^h w$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{m} x_1 x_j f_{ij} g_i + \sum_{j=1}^{m} x_1 x_j^2 y^{h-1} w$$

$$= \sum_{j=1}^{m} x_1 x_j\left(\sum_{i=1}^{d} f_{ij} g_i + x_j y^{h-1} w\right).$$

Hence by [**4**, Lemma 15] for series we obtain (2).

Note that if in (B) $v=0$ and the monomials of $\sum_{i=1}^{d} f_i g_i$ do not begin with $x_1 x_m^{2k}$ then $u=0$ and also $w=0$. Hence $\sum_{i=1}^{d} f_{ij} g_i \equiv 0$ for $j=1,\cdots,m$.

(C) *If* $\sum_{i=1}^{d} f_i g_i \equiv 0$ *then for* $0 \leqq l < \min\{\beta, k\}$ *there exist* $w_l \in F[[x]]$ *and* p_{il}, $i=1,\cdots,d$, *beginning with* x_1 *such that* $2(\beta-l)=v(p_{1l}) = \min\{v(p_{il})\,|\,i=1,\cdots,d\}$ *and*

$$\sum_{i=1}^{d} p_{il} g_i + x_1 y^{k-l} w_l \equiv 0.$$

Proof. For $l=0$ the result follows by (A) with $p_{i0}=f_i'$, $w_0=0$. Assume the result holds for l such that $l+1<\min\{\beta, k\}$. By (B) with w_l, p_{il}, $\beta-l$, $k-l$, replacing v, f_i, β, h respectively we obtain the result for $l+1$ if we take f_{ij_0}, and by (A) with f_{ij_0} replacing f_i we obtain $p_{i,l+1}$. For $i=1,\cdots,d$ we have $v(p_{i,l+1}) \geqq v(p_{1,l+1}) = 2(\beta-l)-2 = 2(\beta-(l+1))$.

(D) *If* $\sum_{i=1}^{d} f_i g_i \equiv 0$ *with* f_i *special*, $g_i \in \mathcal{P}$ *and* $v(f_i) \geqq v(f_1) = 2\beta > 0$, *then there exist* $t_1,\cdots,t_d \in \mathcal{P}$ *with* $t_1=1$ *such that* $\sum_{i=1}^{d} t_i g_i \equiv 0$.

Proof. First we prove the result for $\beta \leqq k$. By (C) for $l=\beta-1$ we obtain $\sum_{i=1}^{d} p_{i,\beta-1} g_i + x_1 y^{k-(\beta-1)} w_{\beta-1} \equiv 0$ and $v(p_{i,\beta-1}) \geqq v(p_{1,\beta-1}) = 2$. Now denote $p_{i,\beta-1}$ by q_i and use (B) for q_i replacing f_i. Then

$$\sum_{i=1}^{d} q_{ij} g_i + x_j y^{k-\beta} w_\beta \equiv 0, \qquad j = 1,\cdots,m, \tag{3}$$

and $v(q_{1j_0})=0$. Hence $q_{1j_0}^{(0)}=$ the constant term of q_{1j_0} is $\neq 0$ and it is therefore 1. Consider the m vectors $(q_{1j}^{(0)},\cdots,q_{dj}^{(0)}) \in F^{(d)}$. Since $d \leqq m-1$ these vectors are dependent over F, hence a sum of some of them is 0. Denote this sum by $\sum'$ and by summation of (3) over those indices which appear in $\sum'$ we obtain

$$\sum_{i=1}^{d} \sum{}' q_{ij} g_i + \sum{}' x_j y^{k-\beta} w_\beta \equiv 0. \tag{4}$$

Since the constant term of $\sum' q_{ij}$ is $\sum' q_{ij}^{(0)} = 0$ for $i = 1, \cdots, d$, it follows that we can write $\sum' q_{ij} = \sum_{l=1}^{m} x_l r_{il}$. Let j_1 be an index which occurs in $\sum'$, then we have by (4) and by [4, Lemma 5] for series $\sum_{i=1}^{d} x_{j_1} r_{ij_1} g_i + x_{j_1} y^{k-\beta} w_\beta \equiv 0$ and hence also

$$\sum_{i=1}^{d} x_{j_0} r_{ij_1} g_i + x_{j_0} y^{k-\beta} w_\beta \equiv 0. \tag{5}$$

Now adding this with (3) for $j = j_0$ we get

$$\sum_{i=1}^{d} (q_{ij_0} + x_{j_0} r_{ij_1}) g_i \equiv 0. \tag{6}$$

Since the constant term of $q_{1j_0} + x_{j_0} r_{1j_1}$ is $q_{1j_0}^{(0)} = 1$ it follows that $q_{1j_0} + x_{j_0} r_{1j_1}$ is invertible in $\mathcal{P}$ and multiplying (6) by $(q_{1j_0} + x_{j_0} r_{1j_1})^{-1}$ we obtain $\sum_{i=1}^{d} t_i g_i$ with $t_i \in \mathcal{P}$ and $t_1 = 1$. This is the desired result for $\beta \leqq k$.

If $\beta > k$ then $v(f_i) \geqq 2\beta \geqq 2k+2$ and $f_1, \cdots, f_d$ being special have not monomials which begin with $x_1 x_m^{2k}$. Hence by the remark at the end of (B) we have $\sum_{i=1}^{d} f_{ij_0} g_i \equiv 0$. But $v(f_{ij_0}) \geqq v(f_{1j_0}) = 2(\beta - 1)$ and the result follows by induction. This completes the proof of (D) and the assertion of Theorem 4 follows.

Now we can obtain our main result which is

THEOREM 5. *Each of the sequences* $\{\mathfrak{L}_m\}$, $\{\mathfrak{F}_m\}$ *and* $\{\mathfrak{N}_m\}$ *is strictly decreasing.*

PROOF. We know that these sequences are decreasing. In addition by the previous theorem $\mathcal{R} \in \mathfrak{L}_{m-1}$ and hence by Theorem 3 $\mathcal{R} \in \mathfrak{F}_{m-1}$ (also if $m = 2$ since $1 \in \mathcal{R}$) and $\mathcal{R} \in \mathfrak{N}_{m-1}$. Now we also know that $\mathcal{R} \notin \mathfrak{N}_m$, hence again by Theorem 3 we have $\mathcal{R} \notin \mathfrak{F}_m$ and $\mathcal{R} \notin \mathfrak{L}_m$ and this proves our theorem.

REFERENCES

1. G. M. Bergman, *Commuting elements in free algebras*, Doctoral Dissertation, Harvard University, Cambridge, Mass., 1968.

2. P. M. Cohn, *Free ideal rings*, J. Algebra **1** (1964), 47–69. MR **28** #5095.

3. R. Doss, *Sur l'immersion d'un semi-groupe dans un groupe*, Bull. Sci. Math. (2) **72** (1948), 139–150. MR **10**, 591.

4. A. A. Klein, *Rings nonembeddable in fields with multiplicative semigroups embeddable in groups*, J. Algebra **7** (1967), 100–125. MR **37** #6309.

5. ———, *Necessary conditions for embedding rings into fields*, Trans. Amer. Math. Soc. **137** (1969), 141–151. MR **38** #4510.

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