ON THE EXISTENCE OF INCOMPRESSIBLE SURFACES IN CERTAIN 3-MANIFOLDS. II

WOLFGANG HEIL

We are concerned with the following question: If M is the closure of the complement of a regular neighborhood of a knot in S^3 and if $\pi_1(M)$ contains the fundamental group $\mathfrak{F}$ of a closed (orientable) surface F of genus g, does there exist a nonsingular surface F of genus g in M such that $i_*\pi_1(F)=\mathfrak{F}$? If this is the case we say that $\mathfrak{F}$ is carried by F. In this note we show that if $\mathfrak{F}$ is a normal subgroup of $\pi_1(M)$ then $\mathfrak{F}$ is not carried by a surface $F\subset M$.

In [**5**] Waldhausen gave an example of an irreducible (orientable) 3-manifold N such that $\pi_1(N)$ contains the fundamental group of a torus which is not carried by a nonsingular torus (in fact this N does not contain any incompressible closed orientable surface). The referee also pointed out a paper by Feustel and Max [**7**], in which it is shown that if M is a complement of a nonprime knot a fundamental group of a torus need not be carried by a torus.

We are in the piecewise linear category. The notation and definitions are those of [**3**].

1. Manifolds having the fundamental group of a surface.

PROPOSITION 1. *Let M be a P^2-irreducible 3-manifold. If $\pi_1(M)=\mathfrak{F}$, where $\mathfrak{F}$ is the fundamental group of a closed surface $F\neq S^2$, P^2, then M is a line bundle over F.*

PROOF. (a) We show that $\partial M\neq\emptyset$:

Represent $\pi_1(M)$ as $F_1 * \boldsymbol{Z} F_2$, where F_1 and F_2 are free groups (e.g. $F_1=\{a_1, b_1, \cdots, a_{g-1}, b_{g-1}\}$, $F_2=\{a_g, b_g\}$, $F_1\supset\{\prod_{i=1}^{g-1}[a_i, b_i]\}\approx\boldsymbol{Z}\approx\{[a_g, b_g]\}\subset F_2$). Then by a well-known construction (see [**5**]; the proof of Satz 1.2(2) holds also in case that M is nonorientable and P^2-irreducible) we find an incompressible surface G in M, $\partial G=\partial M\cap G$, such that $i_*\pi_1(G)$ is conjugate in $\pi_1(M)$ to a subgroup of $\boldsymbol{Z}$ (where $i: G\to M$ is inclusion). If M were closed, G would be closed, which is impossible.

(b) We claim that every boundary component of M is incompressible.

For, suppose $G\in\partial M$ is compressible, then by the loop theorem and Dehn lemma [**4**] we find a disc $D\subset M$, $\partial M\cap D=\partial D\not\simeq 0$ on G. Cutting M along D we get one component M' or two components M_1, M_2 and

Received by the editors June 2, 1969.

such that either $\pi_1(M) \approx \pi_1(M') * \boldsymbol{Z}$ or $\pi_1(M) \approx \pi_1(M_1) * \pi_1(M_2)$. Since $\mathfrak{F} = \pi_1(M)$ is not a free product and not cyclic, the first case can not occur and in the second case we have, say $\pi_1(M_1) = 1$. Since M_1 is irreducible, it follows that M_1 is a ball and hence $\partial D \simeq 0$ on G, a contradiction.

(c) Let G be a boundary component of M. Then $i_*\pi_1(G) \subset \pi_1(M)$ (where $i: G \to M$ is inclusion) is a subgroup of $\mathfrak{F}$. Since G is compact, this subgroup is of finite index in $\mathfrak{F}$. (For let $p: \tilde{G} \to F$ be the covering of F for which $p_*\pi_1(\tilde{G}) = i_*\pi_1(G)$. If $p_*\pi_1(\tilde{G})$ were not of finite index in $\mathfrak{F}$, then $\tilde{G}$ would not be compact and $\pi_1(\tilde{G})$ would be a free group.) Consider the lifting $\tilde{M}$ of M which is associated to $i_*\pi_1(G)$. $\tilde{M}$ is compact, irreducible (see the argument in [3]; proof of Theorem 2), and contains a closed incompressible boundary component G' such that $\pi_1(\tilde{M}) = \pi_1(G')$. If G' would be the only boundary component, then the double $D(M)$ would be a closed irreducible manifold with $\pi_1(D(M)) \approx \pi_1(G')$, the fundamental group of a closed surface, which is impossible by (a). Hence $\tilde{M}$ has a (compact) boundary component $G'' \neq G'$. Now it follows from Lemma 5.1 of [6] (see [2, Proposition 5] in the nonorientable case) that $\tilde{M} = G' \times I$. Hence the covering $\tilde{M} \to M$ is one or two-sheeted and M is a line bundle over a closed surface by [6, 4.1] [2, Proposition 4].

REMARK. A somewhat related theorem has been proved by E. Brown and R. Crowell [1].

2. **Normal subgroups which are fundamental groups of surfaces.** If F is a surface in M, let i_* denote the homomorphism induced by inclusion.

THEOREM 2. *Let M be a P^2-irreducible 3-manifold. Suppose there exists a 2-sided closed incompressible surface $F \subset M$ such that $i_*\pi_1(F)$ is a normal subgroup in $\pi_1(M)$. Then one of the following cases holds*:

(a) *M is a fibre bundle over S^1 with fibre F.*

(b) *$M \approx F \times I$.*

(c) *M is a twisted line bundle over a closed surface G and F is parallel to ∂M.*

(d) *F separates M into two twisted line bundles of type* (c).

PROOF. If F does not separate M, let $M' = \mathrm{Cl}(M - U(F))$; if F separates M, let M' be a component of $\mathrm{Cl}(M - U(F))$. We have inclusions $F \xrightarrow{j} M' \xrightarrow{k} M$. Since F is incompressible ker $k_* = 1$ and ker $j_* = 1$. Choosing the basepoints for the fundamental groups in a suitable way, we may assume $i_* = k_* j_*$. Since $i_*\pi_1(F)$ is normal in $\pi_1(M)$ it follows that $j_*\pi_1(F)$ is normal in $\pi_1(M')$. Hence the covering

$\tilde{M}'$ of M', which is associated to $j_*\pi_1(F)$, is regular. Let $p: \tilde{M}' \to M'$ be the covering map and let $F' \subset \partial \tilde{M}'$ be a copy over $jF \subset \partial M'$, for which $p|F'$ is a homeomorphism.

If $p: \tilde{M}' \to M'$ is a homeomorphism, then it follows from Proposition 1 that M' is a line bundle over a surface F^* which is homeomorphic to F.

If p is not a homeomorphism, then there exists a component $F'' \in p^{-1}(F)$, $F'' \neq F'$. F'' is a closed surface, since the covering is regular; (otherwise a closed curve l would lift to an arc in F'', but it lifts to a closed curve in F'). Again it follows from [**6**, 5.1 and 4.1], [**2**, Propositions 5 and 4] that $\tilde{M}' \approx F' \times I \approx F \times I$ and M' is a line bundle over a surface G. So either M' is a twisted line bundle over G and $\partial M' = F$ (and $F \to G$ is a 2-sheeted covering) or $M' \approx G \times I$. In the latter case $M' \approx F \times I$, since $F \subset \partial M'$.

There are the following cases to consider:

(i) F does not separate M.

Then there are two copies of F in $\partial M'$ and $M' \approx F \times I$. This is a special case of case (a) of the theorem.

(ii) F separates M into two components M_1, M_2. Let F^* be a surface homeomorphic to F, let G be a surface for which F is a 2-sheeted covering and denote by $LB(H)$ a twisted line bundle over a surface H.

(α) $M_1 \approx F \times I$, $M_2 \approx F \times I$.

(β) $M_1 \approx F \times I$, $M_2 = LB(F^*)$.

(γ) $M_1 \approx F \times I$, $M_2 = LB(G)$.

(δ) $M_1 \approx LB(F^*)$, $M_2 = LB(F^*)$.

(ϵ) $M_1 \approx LB(F^*)$, $M_2 = LB(G)$.

(η) $M_1 \approx LB(G)$, $M_2 = LB(G)$.

Case (α) implies case (b) of the theorem. In cases (β) and (γ) M is obtained from M_2 by attaching a collar (M_1) to ∂M_2, which implies case (c) of the theorem, (in case (β), F is a torus or a Kleinbottle, since $\partial M_2 \approx F$ is a two-sheeted covering of $F^* \approx F$). In cases (δ), (ϵ), (η) M is separated by F into two line bundles, which proves case (d) of the theorem. (In cases (δ) and (ϵ) it follows again that F is a torus or a Kleinbottle.)

COROLLARY. *Let M be the closure of the complement of a regular neighborhood of a knot in S^3. Suppose $\pi_1(M)$ contains the fundamental group $\mathfrak{F}$ of a closed surface as a normal subgroup. Then there is no 2-sided surface $F \subset M$ which carries $\mathfrak{F}$.*

PROOF. By the theorem, M would satisfy (a), (b), (c) or (d). In cases (a) and (d) M would be closed, in case (b) M would have two boundary components, and in case (c) G would be a Kleinbottle (since ∂M is a torus and M is orientable) and $H_1(M)$ would be $\boldsymbol{Z}\times\boldsymbol{Z}_2$, a contradiction.

ADDED IN PROOF. In Theorem 1 of [**3**] we have to add the hypothesis that F has at most one boundary component. William Jaco kindly pointed out counterexamples to this theorem in case that F is a planar surface having more than one boundary component. If F would have more than 1 boundary component then (in the proof of the theorem) we cannot assume that $H^{(1)}$ is a wedge of closed curves and then the map $f|H\times 0\cup\partial H\times I$ can in general not be extended to $H^{(1)}\times I$ such that $f|H^{(1)}\times 1\subset G$.

REFERENCES

1. E. Brown and R. Crowell, *Deformation retractions of* 3-*manifolds into their boundaries*, Ann. of Math. (2) **82** (1965), 445–458. MR **31** #6236.

2. W. Heil, *On P^2-irreducible* 3-*manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775.

3. ———, *On the existence of incompressible surfaces in certain* 3-*manifolds*, Proc. Amer. Math. Soc. **23** (1969), 704–707.

4. J. Stallings, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12–19. MR **22** #12526.

5. F. Waldhausen, *Gruppen mit Zentrum und* 3-*dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517. MR **38** #5223.

6. ———, *On irreducible* 3-*manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR **36** #7146.

7. Charles Feustel and N. Max, *On a problem of R. H. Fox*, (to appear).

RICE UNIVERSITY, HOUSTON, TEXAS 77001