QUASI-COMPACT OPERATORS IN TOPOLOGICAL LINEAR SPACES

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The classical theorems of Riesz [1] on compact operators have been extended by Leray [2] and Williamson [3] to the context of topological linear spaces. Ringrose [4] has shown that if an operator on such a space is compact, the square of its adjoint is also compact, where the topology on the dual space is that of uniform convergence on bounded sets. Thus if an operator is continuous and some power is compact, its adjoint shares the same property. We shall call such operators *quasi-compact*; in this note we prove the Riesz theorems for quasi-compact operators in topological linear spaces. This has already been done for Banach spaces by Zaanen [5, Chapter 11].

"Space" will mean a Hausdorff topological linear space. Most of our definitions are as in Bourbaki [6], [7]. A set E is *circled* if for every complex number ϵ such that $|\epsilon| \leqq 1$, $\epsilon E \subset E$. The circled neighborhoods of 0 form a base of neighborhoods at 0. Hereafter, "neighborhood" will mean "circled neighborhood of 0"; T will be a continuous operator on a space X such that T^r is compact; W will denote an open neighborhood such that the closure $(T^rW)^-$ is compact, and $U = \lambda - T$, where λ is a nonzero scalar.

LEMMA 1 (SEE [2, LEMMA 6.1]). *For any closed subset F of W^-, UF is closed.*

PROOF. Let $y_0 \in (UF)^-$, and let $\mathfrak{B}$ be a base for a filter in UF converging to y_0. Let $U_0 = U|F$; $U_0^{-1}\mathfrak{B}$ is a filter base in F. Let $\mathfrak{B}_1$ be an ultrafilter base refining $U_0^{-1}\mathfrak{B}$. Then $T^r\mathfrak{B}_1$ is an ultrafilter base in $(T^rW)^-$, and by the compactness of this latter set,

$$T^r\mathfrak{B}_1 = (\lambda - U)^r\mathfrak{B}_1 = (\lambda^r - r\lambda^{r-1}U + \cdots + (-1)^rU^r)\mathfrak{B}_1$$

converges to a point z_0 of $(T^rW)^-$.

Now $U\mathfrak{B}_1$ is a filter base refining $U(U_0^{-1}\mathfrak{B}) = \mathfrak{B}$. Since $\mathfrak{B}$ converges to y_0, so does $U\mathfrak{B}_1$. Then

$$(-r\lambda^{-1}U + \cdots + (-1)^r\lambda^{-r}U^r)\mathfrak{B}_1$$

converges to

$$w_0 = (-r\lambda^{-1} + \cdots + (-1)^r\lambda^{-r}U^{r-1})y_0,$$

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and hence $\mathfrak{B}_1$ converges to $x_0=\lambda^{-r}z_0-w_0$, which belongs to the closed set F. Thus $U\mathfrak{B}_1$ converges to Ux_0 and since $U\mathfrak{B}_1$ also converges to y_0, $Ux_0=y_0$ and $y_0\in UF$. This shows that $UF\supset(UF)^-$, so that UF is closed. This completes the proof.

It follows from [2, Lemma 6.2] that UX is closed, and hence for every n, U^nX is closed.

LEMMA 2 (SEE [3, COROLLARY TO LEMMA 5]). *Let Y be a finite-dimensional subspace of X. Then if there exists a positive integer n such that $Y\cap U^nX=\{0\}$, $Y\cap W^-$ is compact.*

PROOF. If $T^ry=0$, where $y\in Y$, we have

$$(\lambda - U)^r y = (\lambda^r y - r\lambda^{r-1}Uy + \cdots + (-1)^r U^r y) = 0$$

so that

$$y = -(-r\lambda^{-1}Uy + \cdots + (-1)^r\lambda^{-r}U^ry).$$

We substitute this expression for y into itself n times, obtaining a polynomial in U with no power of U occurring that is less than n. But $Y\cap U^nX=\{0\}$, so $y=0$ and T^r is one-to-one on Y. From [3, Lemma 5], $Y\cap W^-$ is compact.

LEMMA 3 (SEE [3, LEMMA 2]). *If T is any linear operator on X with continuous inverse and closed range, and if Y is a finite-dimensional subspace such that $Y\cap TX=\{0\}$, then for each neighborhood N there is a neighborhood M such that $TN\supset(Y+M)\cap TX$.*

PROOF. It is well known that the relative topology on the sum of two independent closed subspaces, one of which has finite dimension, is the product topology. For any neighborhood N, TN is a neighborhood in TX. Thus there exists a neighborhood M in X such that $TN+Y\supset M\cap(TX+Y)$, since Y is open in itself. Then it follows that $TN\supset(M+Y)\cap TX$.

THEOREM 1 (SEE [3, THEOREM 1]). *Either $U=\lambda-T$ is a homeomorphism of X onto X or U is not one-to-one.*

PROOF. If U is not a homeomorphism, exactly one of the following cases holds:

1. U is not one-to-one.
2. U is one-to-one but its inverse is not continuous.
3. U is one-to-one, has a continuous inverse, but is not onto.

If condition 2 holds, there exists a neighborhood N_1 such that $0\in(U(N_1'))^-$ (the prime denotes complementation); let N_2 be an open neighborhood such that $N_2\subset W\cap N_1$; then $0\in(U(N_2'))^-$.

If B is any neighborhood, $B\cap U(N_2')$ is nonempty and hence there is an element $x\in U^{-1}B\cap N_2'$; by [3, Lemma 3] there exists an $r\in(0,1]$ such that $rx\in 2N_2\sim N_2$. Hence

$$rx \in U^{-1}B \cap N_2' \cap 2N_2 \subset U^{-1}B \cap N_2' \cap 2W.$$

Now let $\mathfrak{B}$ be a base of neighborhoods of 0 and let

$$\mathfrak{B}_1 = \{U^{-1}B \cap N_2' \cap 2W \mid B \in \mathfrak{B}\}.$$

By the previous argument, $\mathfrak{B}_1$ consists of nonempty sets. It is a filter base, and so is $U\mathfrak{B}_1$. Every set of B contains a set of $U\mathfrak{B}_1$, so $U\mathfrak{B}_1$ converges to 0. Also $T^r\mathfrak{B}_1$ has an adherent point x_0 because for all $B_1\in\mathfrak{B}_1$,

$$T^rB_1 = T^r(U^{-1}B \cap N_2' \cap 2W) \subset (T^r(2W))^-,$$

this last set being compact.

Now

$$T^r = \lambda^r - r\lambda^{r-1}U + \cdots + (-1)^rU^r$$

so that

$$(1 - r\lambda^{-1}U + \cdots + (-1)^r\lambda^{-r}U^r)\mathfrak{B}_1$$

has the adherent point $\lambda^{-r}x_0$. Since $U\mathfrak{B}_1$ converges to 0, so does

$$(-r\lambda^{-1}U + \cdots + (-1)^r\lambda^{-r}U^r)\mathfrak{B}_1.$$

By [3, Lemma 4], $\mathfrak{B}_1$ has the adherent point $\lambda^{-r}x_0$. Every set in $\mathfrak{B}_1$ is contained in N_2', so $\lambda^{-r}x_0\in(\lambda^{-r}N_2')^-$ which implies that $\lambda^{-r}x_0=0$. But $\lambda^{-r}Ux_0$ adheres to $U\mathfrak{B}_1$ so that as $U\mathfrak{B}_1$ converges to 0, $\lambda^{-r}Ux_0=0$ and U is not one-to-one. Thus condition 2 does not hold.

If condition 3 holds, let $y\in(UX)'$ and $Y_n=[y,\cdots,U^{n-1}y]$. By [3, Lemma 1], Y_n has dimension n, and $Y_n\cap U^nX=\{0\}$. By Lemma 2, $Y_n\cap W^-$ is compact for all integers n.

Now U^rX is closed and U^r has a continuous inverse, so by Lemma 3, there exists a neighborhood M such that $U^rW\supset(Y_r+M)\cap U^rX$. Let N be an open neighborhood such that $N^-\subset W\cap M$. Then $Y_n\cap N^-$ is compact, and by [2, Lemma 4.1], for each n there exists an element $y_n\in Y_n\cap N^-$ such that $y_n\notin Y_{n-1}+N$.

Let $n>r$. Then

$$y_n = a_0y + a_1Uy + \cdots + a_{r-1}U^{r-1}y + U^rz_{n-1},$$

where $z_{n-1}\in Y_{n-r}$; thus

$$U^r z_{n-1} = y_n - (a_0 y + a_1 U y + \cdots + U^{r-1} y) \in N^- + Y_r \subset M + Y_r$$

so that $z_{n-1} \in W$ for all n. Also $z_{n-1} \notin Y_{n-r-1}$; it follows that $T^r z_{n-1} \in Y_n \sim Y_{n-1}$, so that the elements $T^r z_n$ are distinct.

Now let $n > m > r$. Then

$$\begin{aligned} T^r(z_n - z_m) = (-1)^r y_{n+1} &- (a_0 y + \cdots + a_{r-1} U^{r-1} y) \\ &+ (\lambda^r + \cdots + (-1)^{r-1} r \lambda U^{r-1}) z_n \\ &- (\lambda^r + \cdots + (-1)^r U^r) z_m. \end{aligned}$$

All the terms on the right-hand side are in Y_n except for y_{n+1}. Thus if $T^r(z_n - z_m) \in N$, we would have $y_{n+1} \in Y_n + N$ which is impossible. Hence $T^r z_n \notin T^r z_m + N$; since $\{T^r z_n\} \subset (T^r W)^-$, it follows from [**2**, Lemma 2.1], that $\{T^r z_n\}$ is a finite sequence, which is a contradiction. Theorem 1 is now proved.

It can be shown by minor modifications of the proofs of Leray [**2**, Lemmas 9.1 and 9.2] that $U^{-n}(0)$ is a finite-dimensional subspace of X and $U^{-n}(0) \cap W^-$ is a compact neighborhood in this space.

THEOREM 2. *$U^{-n}(0) \subset U^{-n-1}(0)$ for each integer n; there exists a least integer N such that $U^{-N}(0) = U^{-N-1}(0)$.*

PROOF. If for all n, $U^{-n}(0) \neq U^{-n-1}(0)$, it follows from [**2**, Lemma 4.1], that there exists $x_n \in U^{-n}(0) \cap W^-$ such that $x_n \notin U^{-n+1}(0) + W$.

Let $n > m$; then

$$T(x_n - x_m) = \lambda x_n - (\lambda x_m - U x_m + U x_n).$$

Since $\lambda x_n \in U^{-n+1}(0)$ and $x_n \in U^{-n}(0)$, the second term of the right-hand side belongs to $U^{-n+1}(0)$. Thus

$$\begin{aligned} T(x_n - x_m) &= \lambda(x_n - y_1), \quad \text{where} \quad y_1 \in U^{-n+1}(0), \\ T^2(x_n - x_m) &= \lambda T(x_n - y_1) = \lambda^2(x_n - y_2), \quad \text{where} \quad y_2 \in U^{-n+1}(0), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ T^r(x_n - x_m) &= \lambda^r(x_n - y_r), \quad \text{where} \quad y_r \in U^{-n+1}(0). \end{aligned}$$

But now $x_n - y_r \notin W$, so that $T^r(x_n - x_m) \notin \lambda^r W$. But as $\{T^r x_n\} \subset (T^r W)^-$ it follows from [**2**, Lemma 2.1], that $\{T^r x_n\}$ is a finite sequence. This is a contradiction because the elements $T^r x_n$ are distinct. Theorem 2 is thus proved.

It follows from results of Leray [**2**, Lemmas 9.4 and 9.5], that $U^N X = U^{N+1} X$. That the eigenvalues of a quasi-compact operator on a topological linear space have only 0 as a limit point can be seen by following the methods of Zaanen [**5**, Chapter 11].

REFERENCES

1. F. Riesz, *Über linearen Funktionalgleichungen*, Acta Math. **41** (1918), 71–98.

2. J. Leray, *Valeurs propres et vecteurs propres d'un endomorphisme complètement continu d'un espace vectoriel à voisinages convexes*, Acta Sci. Math. Szeged **12** (1950), pars B, 177–186. MR **12**, 32.

3. J. H. Williamson, *Compact linear operators in linear topological spaces*, J. London Math. Soc. **29** (1954), 149–156. MR **15**, 801.

4. J. R. Ringrose, *Complete continuity conditions on linear operators*, J. London Math. Soc. (3) **8** (1958), 343–356. MR **20** #2621.

5. A. C. Zaanen, *Linear analysis, Measure and integral, Banach and Hilbert space, linear integral equations*, Interscience, New York, North-Holland, Amsterdam and Noordhoff, Groningen, 1953. MR **15**, 878.

6. N. Bourbaki, *Livre* III: *Topologie générale*, Actualités Sci. Indust., no 858, Hermann, Paris, 1940. MR **3**, 55.

7. ———, *Livre* V: *Espaces vectoriels topologiques*, Actualités Sci. Indust., nos. 1189; 1229, Hermann, Paris, 1953; 1955. MR **14**, 880; MR **17**, 1109.

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