# ON $4 / n=1 / x+1 / y+1 / z$ <br> WILLIAM A. WEBB 


#### Abstract

It is shown that the number of positive integers $n \leqq N$ for which $4 / n=1 / x+1 / y+1 / z$ is not solvable in positive integers, is less than a constant times $N /(\log N)^{7 / 4}$.


I. Introduction. Erdös has conjectured that the equation

$$
\begin{equation*}
4 / n=1 / x+1 / y+1 / z \tag{I}
\end{equation*}
$$

is solvable in positive integers for all integers $n \geqq 2$. This has not as yet been proved, but it is known that (I) is solvable for $n$ less than some constant [1], [2], [5]. Using the methods found in these works and some fairly advanced analytic techniques, it can be shown that $S(N) \ll N /(\log N)^{\alpha}$ where $\alpha$ is a constant less than one, and $S(N)$ is the number of positive integers $n$ less than $N$ for which (I) is not solvable. In this paper it is shown that better estimates can be obtained using methods which are essentially elementary.
II. Principal results. By looking at the problem somewhat differently, we are able to obtain various conditions on $n$ which imply the solvability of (I), and then apply sieve methods to obtain an upper bound for $S(N)$. The basic lemma needed is the following:

Lemma 1. $a / b=1 / x+1 / y$ if and only if there exist divisors $d_{1}$ and $d_{2}$ of $b$ such that $a \mid\left(d_{1}+d_{2}\right)(a, b, x, y$ positive integers).
A proof of a generalized form of the above lemma may be found in [6]. We will illustrate the method by considering primes modulo 8 .

Lemma 2. Let $p$ be a prime, then (I) is solvable if:
(i) $p \equiv 7(\bmod 8)$ and $n \equiv 0(\bmod p)$ or $n+1 \equiv 0(\bmod p)$ or $n+2 \equiv 0$ $(\bmod p)$ or $2 n+1 \equiv 0(\bmod p)$ or
(ii) $p \equiv 3(\bmod 8)$ and $n \equiv 0(\bmod p)$ or $n+1 \equiv 0(\bmod p)$ or
(iii) $p \equiv 5(\bmod 8)$ and $n \equiv 0(\bmod p)$.

Proof.
Case (i). Let $p=8 t+7$, and $r=2(t+1)$. Then

$$
4 / n=1 / r n+p / 2(t+1) n .
$$

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If $p \mid n$, the last fraction is reducible and (I) is solvable trivially. (Note: $1 / x=1 /(x+1)+1 / x(x+1)$.) To obtain the other conditions, apply Lemma 1 to the following pairs of divisors of $2(t+1) n: n$ and $1, n$ and $2,2 n$ and 1.

Case (ii). Let $p=4 t+3$, and $r=t+1$. Then

$$
4 / n=1 / r n+p /(t+1) n
$$

If $p \mid n, p /(t+1) n$ is reducible; and if $p \mid n+1$ apply Lemma 1.
Case (iii). If $p \equiv 5(\bmod 8), p+1 \equiv 6(\bmod 8)$ which implies that $p+1$ has a prime divisor $q$ such that $q=4 r-1$. Then

$$
4 / n=1 / r n+q / r n
$$

If $p \mid n$, then $q \mid p+1$ and both $p$ and 1 divide the denominator of the last fraction. Therefore we may apply Lemma 1 again.

Theorem 1. $S(N) \ll N /(\log N)^{7 / 4}$.
Proof. We apply Selberg's sieve to the positive integers $\leqq N$, where the sifting classes for a given prime are those given for $n$ in the statement of Lemma 2. (Note that these residue classes are distinct.)

In particular, we apply Theorem 3, p. 213 of [3], which states that

$$
S(N) \leqq N Q^{-1}+z \prod_{p \in \mathcal{Q}}\left(1-\frac{1}{f(p)}\right)^{-2}
$$

where

$$
\begin{aligned}
\mathcal{P} & =\{p \mid p \text { is a prime } \leqq N, p \not \equiv 1(\bmod 8), p \neq 2\} \\
Q & =\{n \mid n \text { is a positive integer } \leqq N\} \\
f(p) & =p / 4 \quad \text { if } p \equiv 7(\bmod 8) \\
& =p / 2 \quad \text { if } p \equiv 3(\bmod 8) \\
& =p \quad \text { if } p \equiv 5(\bmod 8) \\
\Pi(\mathcal{P}) & =\prod_{p \in \mathcal{P}} p, \\
f(d) & =\prod_{p \mid d} f(p) \quad \text { for } d \mid \prod(P) \\
\mathscr{D} & =\left\{d \mid d \text { divides } \prod(\mathcal{P}), d \leqq z^{1 / 2}\right\} \\
z & =N^{2 / 3}, \\
Q & =Q(\mathbb{D})=\sum_{d \in \mathscr{D}} \frac{1}{g(d)},
\end{aligned}
$$

and

$$
g(d)=f(d) \prod_{p \mid d}\left(1-\frac{1}{f(p)}\right)
$$

provided $\left|R_{d}\right| \leqq d / f(d)$ where

$$
\sum_{n \in \mathbb{Q} ; d \mid \sigma(n)} 1=N / f(d)+R_{d}
$$

and $\sigma(n)=\prod p_{i}$ where the product is over all primes $p_{i}$ which are moduli of sifting classes containing $n$.

If $d=p_{1} p_{2} \cdots p_{r}$, then

$$
\sum_{n \in \mathbb{Q} ; d \mid \sigma(n)} 1=\text { number of } n \leqq N \text { which are sifted by }
$$

$$
p_{1}, p_{2}, \cdots, \text { and } p_{r}
$$

$=$ number of $n \leqq N$ which satisfy a system of
congruences:
(II)

$$
\begin{aligned}
& n \equiv h_{1}\left(\bmod p_{1}\right) \\
& n \equiv h_{2}\left(\bmod p_{2}\right) \\
& \vdots \\
& n \equiv \dot{h}_{r}\left(\bmod p_{r}\right)
\end{aligned}
$$

where $h_{i}$ is any one of the $p_{i} / f\left(p_{i}\right)$ residue classes sifted by our sieve.
The system (II) is equivalent to a congruence

$$
n \equiv H_{j}(\bmod d)
$$

and there are $\left(p_{1} / f\left(p_{1}\right)\right) \cdots\left(p_{r} / f\left(p_{r}\right)\right)=d / f(d)$ such congruences. For each such congruence there are $\left(N / d+E_{j}\right) n$ which are $\leqq N$ and satisfy the congruence, and $\left|E_{j}\right| \leqq 1$. Therefore

$$
\sum_{n \in \mathbb{Q} ; d \mid \sigma(n)} 1=\sum_{j=1}^{d / f(d)}\left(N / d+E_{j}\right)=\frac{N}{f(d)}+R_{d}
$$

where $\left|R_{d}\right|=\left|\sum_{j=1}^{d / f(d)} E_{j}\right| \leqq d / f(d)$.
To complete the proof of Theorem 1, we need only show that

$$
N Q^{-1}+z \prod_{p \in \mathscr{P}}\left(1-\frac{1}{f(p)}\right)^{-2} \ll \frac{N}{(\log N)^{7 / 4}}
$$

$$
\begin{align*}
Q & =Q(D)=\sum_{d \in \mathbb{D}} \frac{1}{g(d)} \geqq \sum_{d \in \mathscr{D}} \frac{1}{f(d)}  \tag{III}\\
& \geqq\left(\sum_{d \leqq N^{1 / 9} ; d \in \mathscr{D}_{5}} \frac{1}{d}\right)\left(\sum_{d \leqq N^{1 / 9} ; d \in \mathscr{D}_{3}} \frac{2^{\Omega(d)}}{d}\right)\left(\sum_{d \leqq N^{1 / 9} ; d \in \mathscr{D}_{7}} \frac{4^{\Omega(d)}}{d}\right)
\end{align*}
$$

where $\mathscr{D}_{j}=\{d \mid d \in \mathscr{D}$ and $p \mid d$ implies $p \equiv j(\bmod 8)\}$ and $\Omega(d)=$ total number of primes dividing $d$. (Since $d$ is square free, $\Omega(d)=\omega(d)$, the number of different primes dividing $d$; but it is convenient to use $\Omega$ rather than $\omega$.)

Hence, we need estimates on sums of the form:

$$
\sum \frac{b^{\Omega(n)}}{n}
$$

To facilitate these estimates, we assume until further notice that the only integers $n$ we deal with have the property that if $p \mid n$ then $p>b$.

Let

$$
T(y)=\sum_{n \leq y} \frac{b^{\Omega(n)}}{n}
$$

where $\sum^{\prime}$ denotes a sum over square free numbers. Also, let

$$
S_{l, x}=\left\{n \leqq x \mid l^{2} \text { is the largest square factor of } n\right\}
$$

Then

$$
\begin{aligned}
\sum_{n \leqq x} \frac{b^{\Omega(n)}}{n} & =\sum_{j=1}^{[\sqrt{ } x]} \sum_{n \in S_{j, x}} \frac{b^{\Omega(n)}}{n}=\sum_{j=1}^{[\sqrt{ } x]} \sum_{n \in S_{1, x} / j^{2}} \frac{b^{\Omega\left(j^{2} n\right)}}{j^{2} n} \\
& =\sum_{j=1}^{[\sqrt{ } x]} \frac{b^{\Omega\left(j^{2}\right)}}{j^{2}} \sum_{n \in S_{1, x / j}^{2}} \frac{b^{\Omega(n)}}{n} \leqq \sum_{j=1}^{[\sqrt{ } x]} \frac{b^{\Omega\left(j^{2}\right)}}{j^{2}} \sum_{n \in S_{1}, x} \frac{b^{\Omega(n)}}{n} \\
& =T(x) \sum_{j=1}^{[\sqrt{ } x]} \frac{\left(b^{2}\right)^{\Omega(j)}}{j^{2}} \leqq T(x) \prod_{b<p \leqq x^{1 / 2}}\left(1+\frac{b^{2}}{p^{2}}+\frac{b^{4}}{p^{4}}+\cdots\right) \\
& \leqq T(x) \prod_{p>b}\left(1+c_{1} \frac{b^{2}}{p^{2}}\right) \leqq c_{2} T(x) .
\end{aligned}
$$

( $c_{i}$ will always denote an unspecified constant.) Hence,

$$
\begin{equation*}
\sum_{n \leqq x}^{\prime} \frac{b^{\Omega(n)}}{n} \geqq c_{3} \sum_{n \leq x} \frac{b^{\Omega(n)}}{n} \tag{IV}
\end{equation*}
$$

Now

$$
\sum_{n \leq y} \frac{1}{n} \geqq c_{4}\left(\sum_{n \leq y} \frac{1}{n}\right)\left(\sum_{j=0}^{\infty} \frac{1}{2^{j}}\right)\left(\sum_{j=0}^{\infty} \frac{1}{3^{j}}\right) \cdots\left(\sum_{j=0}^{\infty} \frac{1}{p_{8}^{j}}\right)
$$

where $p_{s} \leqq b<p_{s+1}$. Therefore

$$
\begin{equation*}
\sum_{n \leqq y} \frac{1}{n} \geqq c_{4} \sum_{m \leqq y} \frac{1}{m} \geqq c_{5} \log y \tag{V}
\end{equation*}
$$

where the last sum is over all positive integers $\leqq y$.
Now

$$
\begin{equation*}
\sum_{n \leqq x} \frac{b^{\Omega(n)}}{n} \geqq \sum_{n \leqq x} \frac{A(n)}{n}=\left(\sum_{n \leqq x^{1 / b}} \frac{1}{n}\right)^{n} \tag{VI}
\end{equation*}
$$

where $A(n)=$ number of ways $n$ can be written as a product of $b$ numbers, each less than $x^{1 / b}$. That $b^{\Omega(n)} \geqq A(n)$ can be seen from the fact that we can assign each prime factor of $n$ to any one of $b$ factors. We get every possible factorization of $n$ in this way, but may get some not counted in $A(n)$. Thus, by (IV), (V), and (VI) we obtain:

$$
\begin{equation*}
\sum_{n \leqq x}^{\prime} \frac{b^{\Omega(n)}}{n} \geqq c_{6}(\log x)^{b} . \tag{VII}
\end{equation*}
$$

Since

$$
\prod_{b<p \leqq x}\left(1-\frac{b}{p}\right)^{-1}<c_{7} \prod_{b<p \leqq x}\left(1-\frac{1}{p}\right)^{-b} \quad[4, \text { Satz 5.5] }
$$

and

$$
\prod_{b<p \leqq x}\left(1-\frac{1}{p}\right)^{-b} \leqq c_{8}(\log x)^{b} \quad[4, \text { Satz 4.1] }
$$

by (VII):
(VIII)

$$
\sum_{n \leq x}^{\prime^{b^{\Omega(n)}}} \frac{c_{9}}{} \prod_{b<p \leq x}\left(1-\frac{b}{p}\right)^{-1} .
$$

Let $L$ be any set of primes, $q$ an element of $L, L^{\prime}=L-\{q\}$, $M_{L}=\{m \mid m$ is a positive integer $\leqq M$, and $p \mid m$ implies $p \in L\}$, and $M_{L^{\prime}}$ defined similarly. We now show

$$
\begin{equation*}
\prod_{p \in L}\left(1-\frac{b}{p}\right)^{-1} \leqq c_{10} \sum_{m \in M_{L}}^{\prime} \frac{b^{\Omega(m)}}{m} \tag{IX}
\end{equation*}
$$

implies

$$
\prod_{p \in L^{\prime}}\left(1-\frac{b}{p}\right)^{-1} \leqq c_{10} \sum_{m \in M_{L^{\prime}}}^{\prime} \frac{b^{\Omega(m)}}{m}
$$

$$
\prod_{p \in L}\left(1-\frac{b}{p}\right)^{-1} \leqq c_{10} \sum_{m \in M_{L}} \frac{b^{\Omega(m)}}{m}
$$

then

$$
\begin{aligned}
\prod_{p \in L^{\prime}}\left(1-\frac{b}{p}\right)^{-1} & =\prod_{p \in L}\left(1-\frac{b}{p}\right)^{-1}\left(1-\frac{b}{q}\right) \\
& \leqq c_{10} \sum_{m \in M_{L}}^{\prime} \frac{b^{\Omega(m)}}{m}\left(1-\frac{b}{q}\right) \\
& =c_{10}\left(\sum_{m \in M_{L}}^{\prime} \frac{b^{\Omega(m)}}{m}-\sum_{m \in M_{L}}^{\prime} \frac{b b^{\Omega(m)}}{q m}\right) \\
& \leqq c_{10}\left(\sum_{m \in M_{L}}^{\prime} \frac{b^{\Omega(m)}}{m}-\sum_{m \in M_{L^{\prime}, q \mid m}^{\prime}}^{\prime} \frac{b^{\Omega(m)}}{m}\right) \\
& =c_{10} \sum_{m \in M_{L^{\prime}}^{\prime}}^{\prime} \frac{b^{\Omega(m)}}{m} .
\end{aligned}
$$

By (VIII) and repeated use of (IX) we have

$$
\begin{equation*}
\sum_{d \leq N^{1 /} / d \in \mathfrak{D}_{j}} \frac{b_{j}^{\Omega(d)}}{d} \geqq c_{9} \prod_{p \leq N^{1 / \vartheta_{;}} ; p \equiv j(\bmod 8)}\left(1-\frac{b_{j}}{p}\right)^{-1} \tag{X}
\end{equation*}
$$

where $b_{3}=2, b_{5}=1$ and $b_{7}=4$. (The condition $p>b_{j}$ is vacuous here.) Now

$$
\begin{gathered}
\log \left(\prod_{p \leq N^{1 / 9} ; p=j(\bmod 8)}\left(1-\frac{b_{j}}{p}\right)^{-1}\right)=-\sum_{p \leq N^{1 / \theta_{;}} ;=j(\bmod 8)} \log \left(1-\frac{b_{j}}{p}\right) \\
\\
\geqq \sum_{p \leqq N^{1 / \theta_{;}} p=j(\bmod 8)} \frac{b_{j}}{p} \geqq \frac{b_{j} \log \log N^{1 / 9}}{\phi(8)}+c_{11} \\
\geqq \frac{b_{j}}{4} \log \log N+c_{12}
\end{gathered}
$$

and therefore

$$
\begin{align*}
\prod_{p \leqq N^{1 /} \theta_{i} p \equiv j(\bmod 8)}\left(1-\frac{b_{j}}{p}\right)^{-1} & \geqq \exp \left(\frac{b_{j}}{4} \log \log N+c_{12}\right)  \tag{XI}\\
& =c_{13}(\log N)^{b_{j} / 4} .
\end{align*}
$$

Hence, by (III), (X), and (XI)

$$
Q(D) \geqq c_{14}(\log N)^{\left(b_{3}+b_{5}+b_{7}\right) / 4}=c_{14}(\log N)^{7 / 4} .
$$

Note that if $d \in \mathscr{D}_{j}$ and $p \mid d$, then $p>b_{j}$. Finally,

$$
\prod_{p \leqq p}\left(1-\frac{1}{f(p)}\right)^{-2} \leqq \prod_{p \leqq N}\left(1-\frac{4}{p}\right)^{-2} \leqq c_{15}(\log N)^{8}
$$

by arguments essentially the same as used above. Therefore

$$
z \prod_{p \in \mathscr{\beta}}\left(1-\frac{1}{f(p)}\right)^{-2} \leqq c_{15} N^{2 / 3}(\log N)^{8}
$$

and so

$$
\begin{aligned}
S(N) & \leqq N \frac{c_{16}}{(\log N)^{7 / 4}}+c_{15} N^{2 / 3}(\log N)^{8} \\
& \leqq c_{17} \frac{N}{(\log N)^{7 / 4}} .
\end{aligned}
$$

This completes the proof of Theorem 1.
III. Concluding remarks. By considering the primes in various residue classes modulo 16 , the results of Theorem 1 can be improved to

$$
S(N) \ll N /(\log N)^{2} .
$$

The exponent of $\log N$ may be improved to $9 / 4-\epsilon$ by considering primes modulo $2^{k}$ for arbitrary $k$ ( $\epsilon$ any small positive number).

The results are still a long way from the conjecture that $S(N)=0$, or even from $S(N) \ll N^{1-\epsilon}$, which would be quite desirable to prove.

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