THE CONNECTION BETWEEN P-FRACTIONS AND ASSOCIATED FRACTIONS

ARNE MAGNUS[1]

ABSTRACT. The associated continued fractions of a power series L is a special case of the P-fraction of a power series L^*. The latter is closely connected with the Padé table of L^*. We prove that every P-fraction is the limit of the appropriate contraction of associated fractions in the sense that as the coefficients of L approach those of L^* the elements and approximants of the contraction approach the elements and approximants of the P-fraction.

Introduction and notation. There is a one-one correspondence between all infinite power series $L=c_0+c_1x+c_2x^2+\cdots$, where

$$\phi_m(L) = \phi_m = \begin{vmatrix} c_1 & c_2 & \cdots & c_m \\ c_2 & c_3 & \cdots & c_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_m & c_{m+1} & \cdots & c_{2m-1} \end{vmatrix} \neq 0, \qquad m = 1, 2, 3, \cdots,$$

and all infinite associated continued fractions

$$A = c_0 + \frac{k_1x}{1+l_1x} - \frac{k_2x^2}{1+l_2x} - \cdots - \frac{k_nx^2}{1+l_nx} - \cdots,$$

with nth approximant K_n/L_n, in the sense that the Taylor expansion of K_n/L_n agrees with L up to and including the term $c_{2n}x^{2n}$. See, for example [1, pp. 128–131]. Note that, by a simple equivalence transformation, A may be written as

$$A = c_0 \; \frac{1}{+\, b_1} \; \frac{1}{+\, b_2} + \cdots \frac{1}{+\, b_n} + \cdots, \tag{1}$$

where $b_n=\alpha_nx^{-1}+\beta_n$, $\alpha_n\neq 0$, $n=1, 2, 3, \cdots$, is a polynomial of degree 1 in x^{-1}. The approximants K_n/L_n are the successive entries down the main diagonal $[n, n]$ of the Padé table of L, [1, p. 260]. A similar correspondence holds between finite or infinite series L representing rational functions and finite associated fractions.

P-fractions, introduced in [2], are continued fractions of the form

Received by the editors October 24, 1969.

AMS Subject Classifications. Primary 4012.

Key Words and Phrases. Continued fractions, associated continued fractions, P-fractions, Padé table.

[1] This research was sponsored by the National Science Foundation under Grant No. GP 9009.

$$P = b_0 \frac{1}{+ b_1} \frac{1}{+ b_2} + \cdots + \frac{1}{b_n} + \cdots,$$

where b_n, $n=0, 1, 2, \cdots$, is a polynomial of degree $N_n \geqq 1$ in x^{-1}. They generalize the associated fractions which occur when $N_n=1$, $n=1, 2, 3, \cdots$, and $b_0 \equiv c_0$. There is a one-one correspondence between all Laurent series

$$\sum_{k=-N_0}^{\infty} d_k x^k = L^*$$

with finite principal part and all P-fractions, in the sense that the nth approximant A_n/B_n of P agrees with L^* up to and including the term of degree $2N_1+2N_2+ \cdots +2N_n+N_{n+1}-1$. P is finite if and only if L^* represents a rational function. We assume here that $N_0=0$. Note that no restriction need be placed on the d_n's to insure the existence of P. The degree N_n of b_n equals $M(n)-M(n-1)$, where $M(n)$ is the index of the nth nonzero $\phi_m^* = \phi_m(L^*)$ and $M(0)=0$, [**3**, p. 216]. If, in particular, $\phi_m^* \neq 0$ for all m, then $N_n=1$, $n=1, 2, 3, \cdots$, which is the case of the associated fraction.

We shall show that every P-fraction is the limit of the appropriate contractions of associated fractions as the c_n's of L approach the d_n's of L^*. Given $\epsilon>0$ and $L^*=d_0+d_1x+d_2x^2+ \cdots$ then there exist $L=c_0+c_1x+c_2x^2+ \cdots$ so that

$$(2) \qquad |c_n - d_n| < \epsilon, \qquad n = 0, 1, 2, \cdots \quad \text{and} \quad \phi_m \neq 0, \quad m = 1, 2, 3, \cdots.$$

That is, L^* can be approximated by a series L which has an associated fraction. The proof is by induction on m. Set $c_0=d_0$ and pick $\phi_1=c_1 \neq 0$ so that $|c_1-d_1|<\epsilon$. Assume $c_0, c_1, \cdots, c_{2n-1}$ have been found so that (2) is satisfied with $m \leqq n$. Observe that $\phi_{n+1}=\phi_n \cdot c_{2n+1} +\psi_n$, where ψ_n is independent of c_{2n+1}. Set $c_{2n}=d_{2n}$ and then pick c_{2n+1} so that $|c_{2n+1}-d_{2n+1}|<\epsilon$ and $\phi_{n+1}\neq 0$, which clearly may be done since $\phi_n \neq 0$.

The theorem. *Given* $L^*=d_0+d_1x+d_2x^2+ \cdots$, *where*

$$\phi_m^* = \begin{vmatrix} d_1 & \cdots & d_m \\ \cdot & \cdots & \cdot \\ d_m & \cdots & d_{2m-1} \end{vmatrix} \neq 0$$

if and only if $m=M(n)$, $n=1, 2, 3, \cdots$, $M(1)<M(2)< \cdots$. *Let* L *be an approximant of* L^* *satisfying* (2) *with the associated fraction* A, *and let* K *be that contraction of* A *which has approximants* $K_{M(n)}/L_{M(n)}$, $n=0, 1, 2, \cdots$, $M(0)=0$. *Then* P *is the limit of* K *in the sense that as*

$\epsilon \to 0$, $K_{M(n)}/L_{M(n)} \to A_n/B_n$, the nth approximant of P. By an equivalence transformation K may be written in such a form that its elements approach those of P.

Proof. We write A in the form (1) so that K_n and L_n are polynomials in x^{-1}, L_n of degree n exactly. By [1, p. 1]

$$\begin{aligned} D_n &= K_{M(n)}L_{M(n-1)} - K_{M(n-1)}L_{M(n)} \\ &= (-1)^{M(n-1)}L_{M(n)-M(n-1)-1,M(n-1)+1}, \end{aligned}$$

is different from zero since the right-hand side is a polynomial in x^{-1} of degree $M(n)-M(n-1)-1$ exactly. This, in turn, insures the existence of the contraction

$$K = t_0 + \frac{s_1}{t_1} + \frac{s_2}{t_2} + \cdots$$

of A with approximants $K_{M(n)}/L_{M(n)}$, $n=0, 1, 2, \cdots$, [1, p. 12]. The elements s_n and t_n of K are given by fractions whose denominators are the D_n's, [1, p. 11]. We now set

$$L_m = q_{0,m}x^{-m} + \cdots + q_{m-1,m}x^{-1} + q_{m,m}$$

and use the fact that

$$x^mL_mL - x^mK_m = (x^{2m+1})$$

to find the following system of equations,

$$\begin{array}{l} q_{m,m}c_1 + q_{m-1,m}c_2 + \cdots + q_{1,m}c_m = -q_{0,m}c_{m+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_{m,m}c_m + q_{m-1,m}c_{m+1} + \cdots + q_{1,m}c_{2m-1} = -q_{0,m}c_{2m}. \end{array}$$

Solving for $q_{i,m}$, $i=1, 2, \cdots, m$, we find

$$q_{i,m} = \phi_m^{-1} q_{0,m} \begin{vmatrix} c_1 & \cdots & c_{m+1} & \cdots & c_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_m & \cdots & c_{2m} & \cdots & c_{2m-1} \end{vmatrix},$$

where the column headed by c_{m+1} replaces that column (in ϕ_m) which is headed by c_{m+1-i}. It is easily seen that $q_{0,m}=\alpha_1\alpha_2 \cdots \alpha_m$ is the quotient of products of the ϕ's and that $\lim \phi_m = \phi_m^*$. When $m=M(n)$, $n=1, 2, 3, \cdots$ we divide K_m and L_m by $q_{0,m}$ and pass to the limit ($c_n \to d_n$, $n=1, 2, \cdots$) and find $L_m/q_{0,m} \to Q_m \not\equiv 0$, $K_m/q_{0,m} \to P_m$ and $x^mQ_mL^* - x^mP_m = (x^{2m+1})$. The degrees of the polynomials x^mQ_m and x^mP_m are less than or equal to $m=M(n)$. Thus $P_{M(n)}/Q_{M(n)}$ is the $[M(n), M(n)]$ entry in the Padé table of L^* and is therefore the nth approximant of P, [1, Theorems 6, 7, pp. 370–371]. This proves that P is the limit of K.

If, in K, we multiply t_n by $\rho_n = q_{0,M(n-1)}/q_{0,M(n)}$ and s_n by $\rho_{n-1}\rho_n$ we obtain an equivalent fraction in which the new values of s_n and t_n have limits σ_n and τ_n respectively. These are therefore the elements of a continued fraction which is equivalent to P.

BIBLIOGRAPHY

1. O. Perron, *Die Lehre von den Kettenbrüchen*, Dritte, verbesserte und erweiterte Aufl. Band II: Analytisch-funktionentheoretische Kettenbrüche, Teubner Verlagsgesellschaft, Stuttgart, 1957. MR **19**, 25.

2. A. Magnus, *Certain continued fractions associated with the Padé table*, Math. Z. **78** (1962), 361–374. MR **27** #272.

3. ———, *Expansion of power series into P-fractions*, Math. Z. **80** (1962), 209–216. MR **27** #273.

COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521