SUBALGEBRAS OF L_∞ OF THE CIRCLE GROUP

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ABSTRACT. In this note, we prove a theorem about subalgebras of a Banach algebra. Thus a theorem of J. P. Kahane and Y. Katznelson implies a theorem of R. Salem and these theorems imply that a number of subspaces of L_∞ of the circle group are not algebras.

An elementary application of the closed graph theorem and the uniform boundedness principle yields the following theorem:

THEOREM 1. *Let* $(E, \|\ \|)$ *be a normed algebra, A a subspace (not necessarily closed) of E. Let* $\|\ \|_1$ *be a norm on A such that* $\|f\|_1 \geqq K\|f\|$, *and* $(A, \|\ \|_1)$ *be a Banach space. Then if A is an algebra then multiplication is* $\|\ \|_1$ *continuous, i.e., there exists M such that* $\|fg\|_1 \leqq M\|f\|_1\|g\|_1$. *Thus if A is an algebra there exists a* $\|\ \|_2$ *equivalent to* $\|\ \|_1$ *such that* $(A, \|\ \|_2)$ *is a Banach Algebra.*

PROOF. (a) Let $L_f: A \to A$ be defined as follows: $L_f g = fg$. Then L_f is a continuous map. Let $g_n \in A$, $\|g_n - g\|_1 \to 0$ and $\|L_f g_n - h\|_1 \to 0$. This implies that $\|g_n - g\| \to 0$, $\|fg_n - h\| \to 0$ and thus $\|fg_n - fg\| \leqq \|f\|\ \|g_n - g\| \to 0$ and $h = fg = L_f g$. The closed graph theorem implies L_f is continuous.

(b) If $\|f\|_1 \leqq 1$, then $\|L_f g\|_1 = \|fg\|_1 = \|L_g f\|_1 \leqq \|L_g\|_1 \|f\|_1 \leqq \|L_g\|_1$. Therefore by the uniform boundedness principle there exists an M such that $\|L_f\|_1 \leqq M$ if $\|f\|_1 \leqq 1$. Consequently $\|fg\|_1 = \|f/\|f\|_1 g\|f\|_1\| = L_{f/\|f\|_1}(g\|f\|_1) \leqq M\|f\|_1\|g\|_1$ and this completes the proof of the theorem.

We note that with an appropriate modification of the proof one can prove a similar theorem to the one above for some Banach spaces E (for example, $L_p(X, \Sigma, \mu)$ where (X, Σ, μ) is a measure space).

Let $L_\infty = L_\infty(\Gamma)$ be the Banach algebra of bounded measurable functions on Γ with norm $\|f\|_\infty = \sup_{t \in \Gamma} |f(t)|$ and $C = C(\Gamma)$ is the closed subspace of L_∞ of continuous functions on Γ. We set

$$S(f) \sim \sum_{-\infty}^{\infty} \hat{f}_n e^{int} \quad \text{the Fourier series of } f;$$

$$S_N(f, t) = \sum_{-N}^{N} \hat{f}_n e^{int} \quad \text{the “symmetric Fourier sum” of } f;$$

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$$S_{MN}(f, t) = \sum_{M}^{N} \hat{f}_n e^{int} \quad \text{the "nonsymmetric sum" of } f.$$

With the norm $\|f\|_U = \sup_N \|S_N(f, \cdot)\|_\infty$, we consider the following subspaces of L_∞:

(a) U_0 is the set of f such that $\|f\|_U < +\infty$.

(b) U is the set of f whose Fourier series is uniformly convergent.

(c) $U_0 \cap C > U$.

With the norm $\|f\|_{U^*} = \sup_{M,N} \|S_{MN}(f, \cdot)\|_\infty$ we consider the following subspaces of L_∞:

(a) U_0^* is the set of f such that $\|f\|_{U^*} < +\infty$;

(b) U^* is the set of f whose nonsymmetric Fourier sum converges uniformly;

(c) U_0^T is the closed subspace of U_0^* (or U_0) which are Taylor series, i.e., $\hat{f}_n = 0$ for $n < 0$;

(d) $U^T = U_0^T \cap U^*$ (or $U_0^T \cap U$);

(e) $U_0^* \cap C$;

(f) $U_0^T \cap C$;

(g) $U^T \cap C$.

We observe that all of these spaces with their respective norm are Banach spaces.

In [2], R. Salem proved that U is not an algebra. In [1], J. P. Kahane and Y. Katznelson proved the stronger result that U^* and U^T are not algebras. Using these theorems we are able to prove the following corollaries:

Corollary 1. *Any closed subspace of U_0 which contains* 1, $e^{i\theta}$, *and* $e^{-i\theta}$ *is not an algebra.*

Proof. If it is an algebra then by Theorem 1 there exists an M such that $\|PQ\|_U \leq M \|P\|_U \|Q\|_U$ for trigonometric polynomials P and Q. Since trigonometric polynomials are dense in U, a simple argument would show that U is an algebra. This contradicts Salem's theorem.

Corollary 2. U_0, U, $C \cap U_0$ *are not algebras.*

Corollary 3. *Any closed subspace of U_0^* which contains* 1 *and* $z = e^{i\theta}$ *is not an algebra.*

Proof. We observe that if P is a trigonometric polynomial then $\|e^{in\theta} P\|_{U^*} = \|P\|_{U^*}$. As in the proof of Corollary 1, we show that U^* is an algebra which is a contradiction.

Corollary 4. U_0^*, U^*, U_0^T, U^T, $U_0^* \cap C$, $U_0^T \cap C$, and $U^T \cap C$ *are not algebras.*

We conclude with the observation that one can prove that if some closed subspace of U_0 which contains 1 and z is not an algebra then none of the subspaces are algebras. This observation could possibly lead to a simpler proof of Kahane and Katznelson's theorem.

References

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