ON THE SOLUTION OF LINEAR FUNCTIONAL EQUATIONS BY AVERAGING ITERATION

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Curtis Outlaw and C. W. Groetsch [4] have recently shown that if T is an asymptotically convergent continuous linear self-mapping of a Banach space E, and if f is in the range of $I-T$, and $0<\lambda<1$, and $V_\lambda=\lambda I+(1-\lambda)(T+f)$, then for each $x\in E$ the sequence $\{V_\lambda^n x\}$ converges to a solution u of the equation $u-Tu=f$. Since under these same hypotheses Browder and Petryshyn [1] showed that the sequence $\{(T+f)^n x\}$ also converges to a solution u of $u-Tu=f$, the Outlaw-Groetsch theorem essentially says that the averaged iteration $x_{n+1}=V_\lambda x_n=\lambda x_n+(1-\lambda)(Tx_n+f)$ yields a conservative process. The purpose of the present paper is to establish some fairly general conditions under which $\{V_\lambda^n x\}$ will converge to a solution u of $u-Tu=f$ (even when $\{(T+f)^n x\}$ does not).

Suppose E is a Banach space, $T:E\to E$ is a continuous linear operator, and $f\in E$. For $0<\lambda<1$ we define $S_\lambda=\lambda I+(1-\lambda)T$, $V_\lambda=\lambda I+(1-\lambda)(T+f)$, and $A^\lambda=[a_{nj}]$ where $a_{11}=1$, $a_{1j}=0$ for $j>1$, and for $n>1$, $a_{nj}=\binom{n-1}{j-1}\lambda^{n-j}(1-\lambda)^{j-1}$ for $1\leqq j\leqq n$, and $a_{nj}=0$ for $j>n$. It is easily seen that A^λ is a lower-triangular, nonnegative, infinite matrix with each row-sum equal to one and each column-limit equal to zero. For $n>1$ we have the real polynomial $a_n^\lambda(t)$ defined by

$$S_\lambda^{n-1}(t)=(\lambda+(1-\lambda)t)^{n-1}=\sum_{j=1}^{n}a_{nj}t^{j-1}=a_n^\lambda(t).$$

So, defining $A_n^\lambda=a_n^\lambda(T)$, we have $S_\lambda^{n-1}=A_n^\lambda$, since T is a linear operator. Defining

$$b_n^\lambda(t)=(1-a_n^\lambda(t))/(1-t),\quad\text{and}\quad B_n^\lambda=b_n^\lambda(T),$$

we have, for $n\geqq 2$, $B_n^\lambda=(1-\lambda)[I+S_\lambda+S_\lambda^2+\cdots+S_\lambda^{n-2}]$, since $I-T=(1-\lambda)^{-1}(I-S_\lambda)$. Also, for $n\geqq 2$, we have

$$\begin{aligned}V_\lambda^{n-1}&=[S_\lambda+(1-\lambda)f]^{n-1}\\&=S_\lambda^{n-1}+(1-\lambda)[I+S_\lambda+\cdots+S_\lambda^{n-2}](f)\\&=A_n^\lambda+B_n^\lambda f,\end{aligned}$$

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since S_λ is linear. It now follows at once from Theorem 3 of [3] that if T is asymptotically A^λ-convergent (i.e., $\{I, T, T^2, \cdots\}$ is an asymptotically convergent semigroup with $\{A_n^\lambda\}$ as a system of almost invariant integrals) then $\{B_n^\lambda\}$ forms a system of companion integrals for $\{A_n^\lambda\}$ with respect to T. Consequently, with the above observation that $V_\lambda^{n-1}x = A_n^\lambda x + B_n^\lambda f$ for all $x \in E$ and all $n \geqq 2$, Theorem 4 of [3] specializes to yield the following result.

THEOREM 1. *Suppose T is an asymptotically A^λ-convergent continuous linear operator on the Banach space E, where $0<\lambda<1$, and suppose $f \in E$. Then, the following are true:*

(a) *If f is in the range of $I-T$, then for any $x \in E$ the sequence $\{V_\lambda^n x\}$ converges to a solution u of the equation $u - Tu = f$.*

(b) *If, for some $x \in E$, $\{V_\lambda^n x\}$ has a subsequence $\{V_\lambda^{n_i} x\}$ which converges weakly to a point $y \in E$, then $y - Ty = f$ and $\{V_\lambda^n x\}$ converges to y.*

(c) *If, for some $x \in E$, $\{V_\lambda^n x\}$ is contained in a weakly compact subset of E, then $\{V_\lambda^n x\}$ converges to a solution of the equation $u - Tu = f$.*

In order to apply Theorem 1, one has to know only that the continuous linear operator T is asymptotically A^λ-convergent for some λ, $0<\lambda<1$. In this direction we have the following result.

THEOREM 2. *Suppose T is a continuous linear operator on a uniformly convex Banach space E, and suppose $\|T\| \leqq 1$. Then for any λ, $0<\lambda<1$, T is asymptotically A^λ-convergent.*

PROOF. By Theorem 5 of [3] it suffices to show that

(a) T is asymptotically A^λ-bounded,

(b) T is asymptotically A^λ-regular, and

(c) $\{A_n^\lambda x\}$ clusters weakly for each $x \in E$.

Since $\|T\| \leqq 1$ we have

$$\|A_n^\lambda\| \leqq \sum_{j=1}^{n} a_{nj} \|T\|^{j-1} \leqq 1$$

for all n, so that (a) is true. To get (b) we first observe that

$$TA_n^\lambda - A_n^\lambda = S_\lambda^{n-1}T - S_\lambda^{n-1} = (1-\lambda)^{-1}[S_\lambda^n - S_\lambda^{n-1}],$$

so that T will be asymptotically A^λ-regular (as defined in [3]) if and only if S_λ is an asymptotically regular operator in the sense of Browder and Petryshyn [2]. Since T is nonexpansive ($\|Tx-Ty\| = \|T(x-y)\| \leqq \|T\| \cdot \|x-y\| \leqq \|x-y\|$) and has at least one fixed point (viz. 0, since T is linear), and since E is a uniformly convex space, Theorem 5 of [2] gives us that for any λ, $0<\lambda<1$, S_λ is an asymptotically regular operator. Hence we get (b). Finally, since uniformly convex Banach spaces are reflexive, closed spheres in E are weakly compact. Since for any $x \in E$ we have for all n

$$\|A_n^\lambda x\| \leqq \|A_n^\lambda\| \cdot \|x\| \leqq \|x\|,$$

it follows that for any $x \in E$ the sequence $\{A_n^\lambda x\}$ clusters weakly; and so we get (c). Q.E.D.

COROLLARY. *Suppose T is a continuous linear operator on a uniformly convex Banach space E, and suppose $\|T\| \leqq 1$. Then for any λ, $0<\lambda<1$, and for any $x \in E$ the sequence $\{S_\lambda^n x\}$ converges (strongly) to a fixed point of T.*

PROOF. By Theorem 2, T is asymptotically A^λ-convergent. Since T is linear, $(I-T)(0)=0$. Hence part (a) of Theorem 1 can be applied, with $f=0$. But for $f=0$ we have $V_\lambda = S_\lambda$, and, of course, solutions of $u-Tu=0$ are fixed points of T. Q.E.D.

REMARK 1. Setting $\lambda=1/2$ in the above corollary provides an affirmative answer to a conjecture of Outlaw and Groetsch [**4**, p. 431].

REMARK 2. It is easily seen that there are continuous linear operators which satisfy the hypotheses of Theorem 2, but which are not asymptotically convergent operators, e.g., any rotation of a finite-dimensional Euclidean space about the origin, or, in l^2, the shift operator $(x_1, x_2, \cdots) \rightarrow (0, x_1, x_2, \cdots)$.

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