ANALYTICITY AND CONTINUATION OF CERTAIN FUNCTIONS OF TWO COMPLEX VARIABLES[1]

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ABSTRACT. This paper shows that the satisfaction of a certain quadratic relation is a sufficient condition that a continuous, symmetric function of two complex variables on a domain be analytic and be continuable to a particular larger domain. This quadratic relation is of the same type as that involved in the Grunsky inequalities.

In proving a generalization of the Grunsky inequalities, Bergman and Schiffer [3] announced a theorem on analytic continuation of a function of two complex variables. In extending the Grunsky inequalities in another way, Alenicyn [1] found this theorem on continuation useful. The purpose of this note is to strengthen the Bergman-Schiffer theorem to be more natural for both applications and to provide a proof that is more direct than the formal computation in the original proof.

Suppose $\mathfrak{D}$ and $\mathcal{G}$ are bounded domains, and $\mathcal{G}$ is contained in $\mathfrak{D}$. Let $\int_{\mathfrak{D}} \cdot dA_z$ denote area integration as z ranges over $\mathfrak{D}$. Let $K_{\mathfrak{D}}(z, \bar{\zeta})$ be the Bergman kernel function [2] for the domain $\mathfrak{D}$.

THEOREM. *If $V(z, \zeta)$ is a symmetric, continuous, complex-valued function on $\mathcal{G} \times \mathcal{G}$,* and

$$(1) \qquad \left| \int_{\mathcal{G}} \int_{\mathcal{G}} V(z, \zeta) \overline{\phi(z)}\ \overline{\phi(\zeta)} dA_z dA_\zeta \right| \leqq \int_{\mathcal{G}} \int_{\mathcal{G}} K_{\mathfrak{D}}(z, \bar{\zeta})\ \overline{\phi(z)} \phi(\zeta) dA_z dA_\zeta$$

for all continuous, complex-valued function ϕ with compact support in $\mathcal{G}$, then $V(z, \zeta)$ is analytic in $\mathcal{G} \times \mathcal{G}$ and can be continued onto $\mathfrak{D} \times \mathfrak{D}$.

PROOF. Let G be a subdomain of $\mathcal{G}$ such that the closure $\bar{G}$ is contained in $\mathcal{G}$. There exists a complete orthonormal system of analytic functions $\{\phi_n\}_{n=1}^{\infty}$ on $\mathfrak{D}$, which is also orthogonal on G, [2]. Let

$$k_n^2 = \int_G \phi_n(z) \overline{\phi_n(z)} dA_z \qquad \text{for } n = 1, 2, \cdots.$$

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Then $\{\phi_n(z)/k_n\}_{n=1}^{\infty}$ is an orthonormal system on G, but is not necessarily a complete system.

Let $\mathfrak{A}_{nm}$ be defined by

$$k_n k_m \mathfrak{A}_{nm} = \int_G \int_G V(z, \zeta)\overline{\phi_n(z)}\ \overline{\phi_m(\zeta)}\, dA_z dA_\zeta. \tag{2}$$

Since $V(z, \zeta)$ is continuous on $\overline{G}\times\overline{G}$, $\int_G\int_G |V(z, \zeta)|^2 dA_z dA_\zeta < \infty$. Then by the usual argument, [2]

$$\sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)$$

converges uniformly on compact subsets of G. It is now shown that the series converges to $V(z, \zeta)$.

Suppose $\Gamma_1(z)$ is a continuous function on G, has $\int_G |\Gamma_1(z)|^2 dA_z < \infty$, and is orthogonal to ϕ_n on G for $n = 1, 2, \cdots$. Let $\Gamma_2(z)$ be any continuous function on $\overline{G}$, and λ be a real number. By using the symmetry of $V(z, \zeta)$,

$$\begin{aligned}\left|\int_G\int_G V(z, \zeta)\overline{[\Gamma_1(z) + \lambda\Gamma_2(z)]}\,\overline{[\Gamma_1(\zeta) + \lambda\Gamma_2(\zeta)]}\, dA_z dA_\zeta\right| \\ = \Bigg|\int_G\int_G V(z, \zeta)\overline{\Gamma_1(z)}\ \overline{\Gamma_1(\zeta)}\, dA_z dA_\zeta \\ + 2\lambda \int_G\int_G V(z, \zeta)\overline{\Gamma_1(z)}\ \overline{\Gamma_2(\zeta)}\, dA_z dA_\zeta \\ + \lambda^2 \int_G\int_G V(z, \zeta)\overline{\Gamma_2(z)}\ \overline{\Gamma_2(\zeta)}\, dA_z dA_\zeta\Bigg|\end{aligned}$$

on the other hand, by (1) and the orthogonality of Γ_1 to ϕ_n on G

$$\leqq \lambda^2 \int_G\int_G K_{\mathfrak{D}}(z, \overline{\zeta})\overline{\Gamma_2(z)}\Gamma_2(\zeta)\, dA_z dA_\zeta \qquad \text{for all real } \lambda.$$

Thus

$$\int_G\int_G V(z, \zeta)\overline{\Gamma_1(z)}\ \overline{\Gamma_1(\zeta)}\, dA_z dA_\zeta = 0$$

and

$$\int_G\int_G V(z, \zeta)\overline{\Gamma_1(z)}\ \overline{\Gamma_2(\zeta)}\, dA_z dA_\zeta = 0. \tag{3}$$

Letting

$$\Gamma_2(\zeta) = \int_G V(z, \zeta)\overline{\Gamma_1(z)}\,dA_z,$$

by (3)

$$\int_G \left\{ \int_G V(z, \zeta)\overline{\Gamma_1(z)} \left[\overline{\int_G V(z, \zeta)\overline{\Gamma_1(z)}\,dA_z} \right] dA_z \right\} dA_\zeta = 0,$$

$$\int_G \left| \int_G V(z, \zeta)\overline{\Gamma_1(z)}\,dA_z \right|^2 dA_\zeta = 0.$$

Hence

$$\int_G V(z, \zeta)\overline{\Gamma_1(z)}\,dA_z = 0 \qquad \text{for all } \zeta \text{ in } G, \tag{4}$$

for all functions $\Gamma_1(z)$ that are continuous on G, have $\int_G |\Gamma_1(z)|^2 dA_z < \infty$ and are orthogonal to $\phi_n(z)$ on G for $n=1, 2, \cdots$.

Let $\delta_k(z-z_0)$ be the kth continuous approximation to the delta function at z_0 such that $\delta_k(z-z_0)=0$ for all z in G with $|z-z_0|>1/k$. Then $\delta_k(z-z_0)$ can be expressed by

$$\Psi_k(z) + \sum_{n=1}^{\infty} b_n^{(k)} \phi_n(z)$$

for z in G, where $\Psi_k(z)$ is orthogonal to $\phi_n(z)$ for $n=1, 2, \cdots$, on G, has $\int_G |\Psi_k(z)|^2 dA_z < \infty$ and is continuous on G. By (4), the definition of $\mathfrak{A}_{nm}$,

$$\int_G \int_G \left[V(z, \zeta) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta) \right] \overline{\left[\Psi_k(z) + \sum_{n=1}^{\infty} b_n^{(k)} \phi_n(z) \right]}$$
$$\cdot \overline{\left[\Psi_k(\zeta) + \sum_{m=1}^{\infty} b_m^{(k)} \phi_m(\zeta) \right]} dA_z dA_\zeta = 0.$$

If z_0 is in G, taking $\lim_{k\to\infty}$ yields

$$V(z_0, z_0) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z_0)\phi_m(z_0) = 0. \tag{5}$$

A similar computation using $\delta_k(z-z_0)+\delta_k(z-z_1)$ for the test function yields

$$V(z_0, z_0) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z_0)\phi_m(z_0) + V(z_0, z_1) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z_0)\phi_m(z_1)$$
$$+ V(z_1, z_0) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z_1)\phi_m(z_0) + V(z_1, z_1) - \sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z_1)\phi_m(z_1) = 0.$$

By (5) and the symmetry of V and thus of $\mathfrak{A}_{nm}$,

$$V(z,\zeta)=\sum_{n,m=1}^{\infty}\mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)\qquad\text{for all } z \text{ and } \zeta \text{ in } G. \tag{6}$$

Hence V is analytic on $\mathcal{G}\times\mathcal{G}$.

It is now shown that the series (6) converge for z and ζ in $\mathfrak{D}$.

Let $\theta_k(z)+\sum_{n=1}^{\infty}C_n^{(k)}\phi_n(z)$ be the representation of the kth continuous approximation to the delta function at z_0 where the representation holds for z in $\mathfrak{D}$, and $\theta_k(z)$ is orthogonal to $\phi_n(z)$ on $\mathfrak{D}$ for $n=1, 2, \cdots$.

$$\begin{aligned}
&\left|\int_{\mathfrak{D}}\int_{\mathfrak{D}}\left[\sum_{n,m=1}^{L}\mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)\right]\overline{\left[\theta_k(z)+\sum_{n=1}^{\infty}C_n^{(k)}\phi_n(z)\right]}\right.\\
&\qquad\qquad\left.\cdot\overline{\left[\theta_k(\zeta)+\sum_{m=1}^{\infty}C_m^{(k)}\phi_n(\zeta)\right]}dA_z dA_\zeta\right|\\
&=\left|\int_{\mathfrak{D}}\int_{\mathfrak{D}}\left[\sum_{n,m=1}^{L}\mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)\right]\overline{\left[\sum_{n=1}^{L}C_n^{(k)}\phi_n(z)\right]}\right.\\
&\qquad\qquad\left.\cdot\overline{\left[\sum_{m=1}^{L}C_m^{(k)}\phi_m(\zeta)\right]}dA_z dA_\zeta\right|\\
&=\left|\int_{G}\int_{G}\left[\sum_{n,m=1}^{L}\mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)\right]\overline{\left[\sum_{n=1}^{L}\frac{C_n^{(k)}}{k_n}\phi_n(z)\right]}\right.\\
&\qquad\qquad\left.\cdot\overline{\left[\sum_{m=1}^{L}\frac{C_m^{(k)}}{k_m}\phi_m(\zeta)\right]}dA_z dA_\zeta\right|\\
&=\left|\int_{G}\int_{G}V(z,\zeta)\overline{\left[\sum_{n=1}^{L}\frac{C_n^{(k)}}{k_n}\phi_n(z)\right]}\,\overline{\left[\sum_{m=1}^{L}\frac{C_n^{(k)}}{k_m}\phi_m(\zeta)\right]}dA_z dA_\zeta\right|\\
&\leqq\int_{G}\int_{G}K_{\mathfrak{D}}(z,\bar{\zeta})\left[\sum_{n=1}^{L}\frac{C_n^{(k)}}{k_n}\phi_n(z)\right]\left[\sum_{m=1}^{L}\frac{C_m^{(k)}}{k_m}\phi_m(\zeta)\right]dA_z dA_\zeta\\
&=\int_{\mathfrak{D}}\int_{\mathfrak{D}}K_{\mathfrak{D}}(z,\bar{\zeta})\overline{\left[\sum_{n=1}^{L}C_n^{(k)}\phi_n(z)\right]}\left[\sum_{m=1}^{L}C_m^{(k)}\phi_m(\zeta)\right]dA_z dA_\zeta\\
&\leqq\int_{\mathfrak{D}}\int_{\mathfrak{D}}K_{\mathfrak{D}}(z,\bar{\zeta})\overline{\left[\theta_k(z)+\sum_{n=1}^{\infty}C_n^{(k)}\phi_n(z)\right]}\\
&\qquad\qquad\cdot\left[\theta_k(\zeta)+\sum_{m=1}^{\infty}C_m^{(k)}\phi_m(\zeta)\right]dA_z dA_\zeta
\end{aligned}$$

taking the $\lim_{k\to\infty}$

$$(7) \qquad K_{\mathfrak{D}}(z_0, \bar{z}_0) \geqq \left| \sum_{n,m=1}^{L} \mathfrak{A}_{nm}\phi_n(z_0)\phi_m(z_0) \right| \qquad \text{for all } L.$$

A similar computation using a representation of an approximation of the delta function at z_0 plus the delta function at z_1 and utilizing (7) yields

$$K_{\mathfrak{D}}(z_0, \bar{z}_0) + \operatorname{Re} K_{\mathfrak{D}}(z_0, \bar{z}_1) + K_{\mathfrak{D}}(z_1, \bar{z}_1) \geqq \left| \sum_{n,m=1}^{L} \mathfrak{A}_{nm}\phi_n(z_0)\phi_m(z_1) \right|.$$

Hence $\{ \sum_{n,m=1}^{L} \mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta) \}_{L=1}^{\infty}$ is a normal family on $\mathfrak{D}\times\mathfrak{D}$. Since it converges to $V(z, \zeta)$ on $G\times G$, $\sum_{n,m=1}^{\infty} \mathfrak{A}_{nm}\phi_n(z)\phi_m(\zeta)$ must converge to an analytic function on $\mathfrak{D}\times\mathfrak{D}$ that is a continuation of $V(z, \zeta)$.

COROLLARY. *If $V(z, \zeta)$ is a symmetric, continuous, complex-valued function on $\mathcal{G}\times\mathcal{G}$, and*

$$\left| \sum_{n=1}^{L} \sum_{m=1}^{L} \alpha_n\alpha_m V(z_n, z_m) \right| \leqq \sum_{n=1}^{L} \sum_{m=1}^{L} \alpha_n\bar{\alpha}_m K_{\mathfrak{D}}(z_n, \overline{z_m})$$

for all complex vectors $(\alpha_1, \alpha_2, \cdots)$, and $(z_1, z_2, \cdots)$ with all z_n in $\mathcal{G}$, then $V(z, \zeta)$ is analytic in $\mathcal{G}\times\mathcal{G}$ and can be continued onto $\mathfrak{D}\times\mathfrak{D}$.

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