OSCILLATION OF SOLUTIONS OF CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF nTH ORDER

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ABSTRACT. Necessary and sufficient conditions are given that all solutions of $y^{(n)}+f(t, y)=0$ which are continuable to infinity are oscillatory in the case n is even and are oscillatory or strongly monotone in the case n is odd. The results generalize to arbitrary n recent results of J. Macki and J. S. W. Wong for the case $n=2$ and include as special cases results of I. Kiguradze, I. Ličko and M. Švec, and Š. Belohorec.

The equation considered in this paper is

$$y^{(n)} + f(t, y) = 0, \tag{1}$$

where $f(t, y)$ is defined in $S=[0, \infty)\times(-\infty, \infty)$. Let F be the family of solutions of (1) which are indefinitely continuable to the right; i.e. if $y(t)\in F$, then there exists $t_0\geqq 0$ such that $y(t)$ exists on $[t_0, \infty)$. A solution $y(t)$ in F is said to be *nonoscillatory* if, for some T sufficiently large, $y(t)$ is always positive or always negative for $t\geqq T$; otherwise a solution in F is *oscillatory*.

The first theorem generalizes to arbitrary $n\geqq 2$ a theorem of Macki and Wong [**6**, Theorem 1] for the second order equation $y''+f(t, y)=0$, giving necessary and sufficient conditions for solutions of (1) in F to be oscillatory. This theorem also generalizes results of Kiguradze [**2**, Theorem 5] and Ličko and Švec [**4**] for the respective special cases $y^{(n)}+yG(y^2, t)=0$, $G(u, t)$ nonnegative and nondecreasing in u, and $y^{(n)}+a(t)y^{\alpha}=0$, $\alpha>1$ and α the ratio of odd integers. The second theorem generalizes results of Ličko and Švec [**4**] and Belohorec [**1**] for the latter equation when $0\leqq\alpha<1$. It also has points of contact with results of Kiguradze [**3**].

Assume for equation (1) that

(i) $f(t, y)$ is continuous in S;

(ii) $a(t)\phi(y)\leqq f(t, y)$ if $y>0$ and $f(t, y)\leqq b(t)\psi(y)$ if $y<0$, $(t, y)\in S$, where

(iii) $a(t)$ and $b(t)$ are nonnegative and locally integrable on $[0,\infty)$ and neither $a(t)$ nor $b(t)$ is identically zero on any subinterval of $[0, \infty)$,

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(iv) $\phi(y)$ and $\psi(y)$ are nondecreasing, and $y\phi(y)>0$ and $y\psi(y)>0$ on $(-\infty, \infty)$ for $y\neq 0$, and

(v) for some $\alpha \geqq 0$,

$$\int_\alpha^\infty \frac{du}{\phi(u)} < \infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{du}{\psi(u)} < \infty.$$

Conditions (i) through (v) guarantee that equation (1) is *strongly nonlinear* [**2**].

THEOREM 1. *If the function* $f(t, y)$ *in* (1) *satisfies* (i)–(v) *and in addition*

$$(2) \qquad \int_0^\infty t^{n-1}a(t)dt = \int_0^\infty t^{n-1}b(t)dt = \infty,$$

then if n *is even, each solution of* (1) *in* F *is oscillatory, while if* n *is odd, each solution in* F *is either oscillatory or it tends monotonically to zero together with all its first* $n-1$ *derivatives.*

For convenience, before proving Theorem 1 the possible behavior of a nonoscillatory solution is summarized in the following two lemmas [**2**, Lemma 1], [**5**, pp. 410, 418–419], the proofs of which are elementary.

LEMMA 1. *Suppose* $f(t)\in C^k[a, \infty)$, $f(t)\geqq 0$ *and* $f^{(k)}(t)$ *is monotone. Then exactly one of the following is true*:

(i) $\lim_{t\to\infty} f^{(k)}(t)=0$,

(ii) $\lim_{t\to\infty} f^{(k)}(t)>0$ *and* $f(t), \cdots, f^{(k-1)}(t)$ *tend to* ∞ *as* $t\to\infty$.

LEMMA 2. *If* $y(t)\in C^n[a, \infty)$, $y(t)\geqq 0$ *and* $y^{(n)}(t)\leqq 0$ *on* $[a, \infty)$, *then exactly one of the following is true*:

(I) $y'(t), \cdots, y^{(n-1)}(t)$ *tend monotonically to zero as* $t\to\infty$,

(II) *there is an odd integer* k, $1\leqq k\leqq n-1$, *such that* $\lim_{t\to\infty} y^{(n-j)}(t)=0$ *for* $1\leqq j\leqq k-1$, $\lim_{t\to\infty} y^{(n-k)}(t)\geqq 0$, $\lim_{t\to\infty} y^{(n-k-1)}(t)>0$ *and* $y(t), y'(t), \cdots, y^{(n-k-2)}(t)$ *tend to* ∞ *as* $t\to\infty$.

Analogous statements can be made if $y(t)\leqq 0$ and $y^{(n)}(t)\geqq 0$ on $[a, \infty)$.

PROOF OF THEOREM 1. Suppose $y(t)$ is a nonoscillatory solution in F, say $y(t)>0$ for $t\geqq T\geqq 0$. From (1),

$$(3) \qquad y^{(n)}(t) = -f(t, y(t)) \leqq -a(t)\phi(y(t)).$$

By Lemma 1, $y^{(n-1)}(t)$ decreases to a nonnegative limit, so from (3),

$$y^{(n-1)}(s) \geqq \int_s^\infty a(u)\phi(y(u))du. \tag{4}$$

Suppose case I of Lemma 2 holds. Then an integration of (4) $n-2$ times from t to ∞ yields

$$(-1)^n y'(t) \geqq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\phi(y(u))du. \tag{5}$$

If n is even, integrating (5) from T to $t \geqq T$,

$$y(t) \geqq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(y(u))du.$$

Since $\phi(u)$ is nondecreasing,

$$\phi(y(t)) \Big/ \phi\left[\int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(y(u))du\right] \geqq 1,$$

so, as in [6],

$$\int_R^S [\phi(v)]^{-1}dv \geqq \int_r^s \frac{(t-T)^{n-1}}{(n-1)!} a(t)dt,$$

where

$$R = \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(y(u))du$$

and

$$S = \int_T^s \frac{(u-T)^{n-1}}{(n-1)!} a(u)\phi(y(u))du.$$

If for some $r \geqq T$, $R \geqq \alpha$, then condition (v) gives a contradiction to condition (2), while if $R < \alpha$ for all $r \geqq T$, then

$$\alpha > R \geqq \phi(y(T)) \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u)du,$$

again in contradiction to condition (2).

If n is odd, then

$$-y'(t) \geqq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\phi(y(u))du \geqq 0, \tag{6}$$

so $y(t)$ decreases to a limit $L \geqq 0$.

Suppose $L>0$. Then integrating (6) from T to ∞,

$$y(T) > y(T) - L \geqq \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)\phi(y)du$$
$$\geqq \phi(L)\int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)du,$$

since $\phi(y)$ is nondecreasing in y. But this implies

$$\int_0^\infty t^{n-1}a(t)dt < \infty.$$

Suppose now that case II of Lemma 2 holds. Proceeding as in case I,

$$y^{(n-k)}(t) \geqq \int_t^\infty \frac{(u-t)^{k-1}}{(k-1)!}\, a(u)\phi(y)du. \tag{7}$$

Since $y^{(j)}(t)$ increases to infinity, $j<n-k-1$, there exists $t_1 \geqq T$ such that $y^{(j)}(t)>0$ for $t \geqq t_1$, $j=0, \cdots, n-k-1$. Integrating (7) from t_1 to $t>t_1$,

$$y^{(n-k+1)}(t) \geqq \int_{t_1}^t \int_s^\infty \frac{(u-s)^{k-1}}{(k-1)!}\, a(u)\phi(y(u))duds$$
$$\geqq \int_t^\infty \frac{(u-t_1)^k - (u-t)^k}{k!}\, a(u)\phi(y)du,$$

so

$$y^{(n-k+1)}(t) > \int_t^\infty \frac{(t-t_1)^k}{k!}\, a(u)\phi(y)du. \tag{8}$$

Integrating (8) from t_1 to t,

$$y^{(n-k+2)}(t) > \int_t^\infty \frac{(t-t_1)^{k+1}}{(k+1)!}\, a(u)\phi(y)du.$$

Proceeding in this fashion,

$$y'(t) > \int_t^\infty \frac{(t-t_1)^{n-2}}{(n-2)!}\, a(u)\phi(y)du, \tag{9}$$

and a final integration from t_1 to t gives

$$y(t) > \int_{t_1}^t \frac{(u-t_1)^{n-1}}{(n-1)!}\, a(u)\phi(y)du.$$

The proof now proceeds as in case I.

Now suppose $y(t)$ is a solution of (1) such that for $t \geqq T$, $y(t)<0$. The proof is the same as the case $y(t)>0$ with $a(t)$ and $\phi(y)$ replaced respectively by $b(t)$ and $\psi(y)$ everywhere and with appropriate changes in the sense of inequalities.

Under the hypotheses of this theorem it is possible to have a nonoscillatory solution which tends monotonically to zero if n is odd and case I of Lemma 2 holds for this solution. For example, for $n=3$ the equation

$$y''' + e^t y^2 \operatorname{sgn} y = 0$$

has the solution $y(t)=e^{-t}$. In this example one can choose $\phi(y)=\psi(y) = y^2 \operatorname{sgn} y$, $\alpha=1$ and $a(t)=b(t)=e^t$.

Note. If $\int_0^\infty t^{n-1}\, a(t)dt$ in (2) is finite and in condition (ii), $a(t)\phi(y) \leqq f(t, y) \leqq b(t)\psi(y)$ simultaneously in S, a solution in F which is nonoscillatory can be constructed exactly as in [6] making use of the integral equation

$$y(t) = 1 + (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))ds,$$

and similarly, if $\int_0^\infty t^{n-1}\, b(t)dt < \infty$, the integral equation

$$y(t) = -1 + (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))ds$$

may be used to construct a nonoscillatory solution.

In the next theorem, condition (v) is changed so that equation (1) includes the special case

$$y^{(n)} + a(t)y^\alpha = 0, \qquad 0 \leqq \alpha < 1,$$

α the ratio of odd integers.

Before stating the theorem the following lemma is given, a proof of which may be found in [3, Lemma 1].

LEMMA 3. *If $y(t)$, $y'(t)$, $\cdots$, $y^{(n-1)}(t)$ are absolutely continuous and of constant sign on the interval $[t_0, \infty)$, and $y^{(n)}(t)y(t) \leqq 0$, then there exists an integer l, $0 \leqq l \leqq n-1$, which is even if n is odd and odd if n is even, so that*

$$|y(t)| \geqq \frac{(t-t_0)^{n-1}}{(n-1)\cdots(n-l)} |y^{(n-1)}(2^{n-l-1}t)|, \qquad t \geqq t_0.$$

THEOREM 2. *Let f satisfy conditions* (i)–(iv) *and*

(vi) *there exist positive constants* λ_0, M, N *and constants* β, γ, *where* $0 \leqq \beta < 1$, $0 \leqq \gamma < 1$, *such that*

$$\begin{aligned} \phi(\lambda y) &\geqq M\lambda^\beta \phi(y), \quad y > 0, \\ \psi(\lambda y) &\leqq N\lambda^\gamma \psi(y), \quad y < 0, \end{aligned} \qquad \lambda \geqq \lambda_0 > 0.$$

Then if

$$\int^\infty t^{(n-1)\beta} a(t)dt = \int^\infty t^{(n-1)\gamma} b(t)dt = +\infty, \tag{10}$$

each solution in F is oscillatory when n is even and each solution in F is either oscillatory or tends to zero together with its first $n-1$ derivatives if n is odd.

PROOF. Suppose that n is even and there exists a nonoscillatory solution $y(t)$ such that $y(t) > 0$ for $t \geqq t_0$. Then by Lemma 2, $y'(t) \geqq 0$ so $y(t)$ is nondecreasing, and $y^{(n)}(t) \leqq 0$ so $y^{(n-1)}(t)$ is nonincreasing and positive on $[t_0, \infty)$. Therefore by Lemma 3,

$$\begin{gathered} y(t) \geqq y(2^{1-n}t) \geqq At^{n-1} y^{(n-1)}(t), \\ t \geqq 2^n t_0 = t_1, \quad \text{where} \quad A = 2^{-n^2}/(n-1)!. \end{gathered} \tag{11}$$

Because of condition (ii), $y(t)$ must satisfy

$$y^{(n)}(t) + a(t)\phi(y) \leqq 0, \tag{12}$$

and since $y(t)$ is nondecreasing, $ky(t) \geqq \lambda_0$ for $k \geqq \lambda_0/y(t_1)$, $t \geqq t_1$, and $\phi(y) \geqq (ky)^\beta \phi(1/k)M$ by (vi).

Therefore, letting $B = k^\beta \phi(1/k)M > 0$, it follows that $y^{(n)}(t) + Ba(t)y^\beta \leqq 0$, $t \geqq t_1$, and so from (11),

$$y^{(n)}(t) + A^\beta B a(t)^{(n-1)\beta} [y^{(n-1)}(t)]^\beta \leqq 0.$$

Dividing by $[y^{(n-1)}(t)]^\beta$ and integrating from t_1 to t,

$$\int_{y^{(n-1)}(t_1)}^{y^{(n-1)}(t)} \frac{dy}{y^\beta} + A^\beta B \int_{t_1}^{t} s^{(n-1)\beta} a(s)ds \leqq 0. \tag{13}$$

But, since

$$0 > \int_{y^{(n-1)}(t_1)}^{y^{(n-1)}(t)} \frac{dy}{y^\beta} \geqq \int_c^0 \frac{dy}{y^\beta}, \qquad 0 < c < \infty,$$

and the latter integral is finite for $\beta < 1$, this gives a contradiction of (13) as $t \to \infty$ if $\int^\infty t^{(n-1)\beta} a(t)dt = +\infty$. Thus $y(t)$ must be oscillatory.

The case where $y(t)<0$ for $t \geqq t_0$ can be handled similarly and yields a contradiction to the fact that $\int^\infty t^{(n-1)\gamma} b(t)dt = +\infty$. The inequalities in (11) and (12) are reversed with $b(t)\psi(y)$ replacing $a(t)\phi(y)$, and the inequality in (13) is in the same direction but with y replaced by $-y$.

If n is odd and $y(t)$ does not approach zero, then $|y^{(n-1)}(t)|$ is still nonincreasing, so that

$$\begin{aligned} |y(t)| &= |y(t)/y(2^{1-n}t)| \cdot |y(2^{1-n}t)| \\ &\geqq \inf_{t \geqq t_0} |y(t)/y(2^{1-n}t)| \, A \, |y^{(n-1)}(t)| \, t^{n-1}, \qquad t \geqq t_1, \end{aligned}$$

hence $|y(t)| \geqq B_1 t^{n-1} |y^{(n-1)}(t)|$ for constant B_1, and the preceding proof again yields a contradiction to the existence of a nonoscillatory solution in class F.

If conditions (ii) and (vi) are extended so that the inequalities there hold for all y, then by modifications of Kiguradze's proofs [**3**, p. 773], [**3**, Lemma 5], it can be shown that all solutions of (1) are extendible to infinity under the conditions of Theorem 2, and if either

$$\int^\infty t^{(n-1)\beta} a(t)dt \quad \text{or} \quad \int^\infty t^{(n-1)\gamma} b(t)dt$$

is finite, a solution $y(t)$ of (1) can be exhibited such that $\lim_{t\to\infty} y^{(n-1)}(t) = C_0 \neq 0$. Hence, if (ii) and (vi) are valid for all y, condition (10) is necessary and sufficient for all solutions of (1) to oscillate if n is even and for each solution either to oscillate or tend to zero if n is odd.

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