EXTENDING HOMEOMORPHISMS IN COMPACTIFICATION OF FRÉCHET SPACES

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ABSTRACT. In the study of extending homeomorphisms in the compactification of Fréchet spaces into the Hilbert cube Q, it is shown that for a given homeomorphism f of s onto itself and a closed subset K of s satisfying property Z in s, there is a homeomorphism g of s onto itself such that $gfg^{-1}|_{g(K)}$ extends to a homeomorphism of Q onto Q. The proof is by employing the rather useful shifting homeomorphism on Q.

1. Let F be a separable infinite-dimensional Fréchet space. Let Q denote the Hilbert cube $\prod_{i=1}^{\infty} J_i$, $J_i=[-1, 1]$, and let $s=\prod_{i=1}^{\infty} \operatorname{Int} J_i$. s is sometimes called the pseudo-interior of Q. It has been shown that there is a compactification μ of F into Q such that $\mu(F)=s$, [4], [5] and [6].

Let N denote the set of integers and for each $i\in N$, let Q_i denote a copy of Q. We now identify Q as $\prod_{i\in N} Q_i$ and s as $\prod_{i\in N} s_i$ where s_i is the pseudo-interior of Q_i. Let p_i denote the point $(0, 0, \cdots)$ of Q_i or s_i and let $p=(0, 0, \cdots)$ of Q or s.

The *shifting* homeomorphism Φ of Q into itself is defined by

$$\Phi(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, x'_{-1}, x'_0, x'_1, \cdots)$$

where x_i, $x'_i \in Q_i$ and $x'_i = x_{i+1}$, for all $i\in N$.

Φ may be regarded as a *universal* β^*-homeomorphism on Q in the sense of the following main theorem.

THEOREM. *For any closed set A in F and any imbedding f of A into F such that $f(A)$ is closed and both A and $f(A)$ have property Z in F, there is an imbedding μ of F into Q such that $\mu(F)=s$ and $\mu f\mu^{-1}|_{\mu(A)} =\Phi|_{\mu(A)}$.*

Following Anderson [2], a closed set K in a space X has *property* Z in X if for each homotopically trivial nonnull open subset U of X, $U\backslash K$ is both nonnull and homotopically trivial. A homeomorphism h of Q onto Q is a *β^*-homeomorphism* if h carries s onto s.

COROLLARY 1. *Let A, B be closed sets with property Z in s and let f be a homeomorphism of A onto B. Then there is a homeomorphism g of s*

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onto itself such that $gfg^{-1}: g(A) \to g(B)$ *extends to a* β^**-homeomorphism of* Q *onto* Q.

COROLLARY 2. *Let* A, B *be closed sets with property* Z *in* s. *Then there is a homeomorphism* h *of* s *onto* s *such that* $\mathrm{cl}(h(A)), \mathrm{cl}(h(B))$ *are homeomorphic (closures are taken with respect to* Q*).*

2. Before we prove the theorem we need the following lemma. Let $X = s_1 \times s_2$ where $s_1 = s = s_2$. For $i = 1, 2$, let p_i denote the point $(0, 0, \cdots)$ of s_i and let π_1 be the projection function of X onto $s_1 \times p_2$.

LEMMA. *Suppose* A, B *are closed subsets of* $X = s_1 \times s_2$ *such that* $A \subset s_1 \times p_2$ *and* $\pi_1|_B$ *is a homeomorphism of* B *onto* A. *Then there is a homeomorphism* h *of* X *onto* X *such that* $h|_B = \pi_1|_B$.

This lemma may be proved by a repeated use of Tietze's Extension Theorem and is similar to Theorem 4.1 in [1]. We omit the proof here.

PROOF OF THE THEOREM. Let g_0 be a homeomorphism of F onto s. Denote $g_0(A)$ by A_0, $g_0(f(A))$ by B_0. Then $f_0 = g_0 f g_0^{-1}|_{A_0}$ is a homeomorphism of A_0 onto B_0. By [2] there is a homeomorphism g_1 of s onto s such that $g_1(A_0 \cup B_0) \subset \bar{s}_0$ where $\bar{s}_0 = \cdots \times p_{-2} \times p_{-1} \times s_0 \times p_1 \times p_2 \times \cdots$ and $g_1(A_0 \cup B_0)$ has property Z in $\bar{s}_0$. Denote $g_1(A_0)$ by A_1 and $g_1(B_0)$ by B_1. Then $f_1 = g_1 f_0 g_1^{-1}|_{A_1}$ is a homeomorphism of A_1 onto B_1. A_1 and B_1 are both closed and have property Z in $\bar{s}_0$. Identifying $\bar{s}_0$ with s_0, by [2] again there is an homeomorphism G of s_0 onto itself such that $G|_{A_1} = f_1$. Let K be the subset of s defined by

$$K = \{\bar{x} = (\cdots, G^{-2}(x), G^{-1}(x), x, G(x), G^2(x), \cdots) \mid x \in s_0 \text{ and } G^i(x) \in s_i\}.$$

Clearly K is closed and the projection $\bar{x} \to x$ is 1-1. By the lemma there is a homeomorphism g_2 of s onto itself such that $g_2(x) = \bar{x}$ for all $x \in \bar{s}_0$. Let $\mu = g_2 g_1 g_0$. Then $\mu f \mu^{-1}|_{\mu(A)}$ is a homeomorphism of $\mu(A)$ onto $\mu(B)$. A simple verification that $\mu f \mu^{-1}|_{\mu(A)} = \Phi|_{\mu(A)}$ completes the proof of the theorem.

THEOREM 1. *Let* X *be a separable complete metric space and let* f *be a homeomorphism of* X *onto itself. Then there is a homeomorphism* μ *of* X *into* Q *such that*

$$\mu f \mu^{-1}|_{\mu(X)} = \Phi|_{\mu(X)}.$$

COROLLARY. *Let* X, f *be as above, then there is a compactification* μ *of* X *into a compact subset* $\mathrm{cl}(\mu(X))$ *of* Q *such that* $\mu f \mu^{-1}|_{\mu(X)}$ *extends to a homeomorphism of* $\mathrm{cl}(\mu(X))$ *onto itself.*

PROOF OF THEOREM 1. By the classical Banach-Mazur-Kuratowski Theorem, X can be regarded as a closed subset of the space C of continuous real valued functions on the closed unit intervals. It is well known that C can be imbedded as a closed subset of s_0. Hence by considering $s_0 \times s_0$, we may assume that X is imbedded as a closed subset of s_0 and has property Z in s_0. The rest of the proof is the same as the theorem.

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