GROUPS WITH AN IRREDUCIBLE CHARACTER OF LARGE DEGREE ARE SOLVABLE

FRANK DeMEYER

ABSTRACT. The degree of an irreducible complex character afforded by a finite group is bounded above by the index of an abelian normal subgroup and by the square root of the index of the center. Whenever a finite group affords an irreducible character whose degree achieves these two upper bounds the group must be solvable.

Let G be a finite group with an irreducible (complex) character ζ. If Z is the center of G it is easy to prove that $\zeta(1)^2 \leqq [G:Z]$ and if A is an abelian normal subgroup of G it is easy to show that $\zeta(1) \leqq [G:A]$ (see pp. 364–365 of [1]). Say the group G admits an irreducible character of large degree if $\zeta(1)^2 = [G:A]^2 = [G:Z]$, that is whenever the two bounds noted above are achieved simultaneously. Such groups arise in the theory of projective representations and the Galois theory in general rings [2]. The purpose of this note is to give proof of the result stated in the title, thus verifying a special case of a conjecture in [2]. Throughout all unexplained terminology is as in [1].

THEOREM. *Let G be a group with center Z and abelian normal subgroup A so that there is an irreducible complex character ζ on G with $\zeta(1)^2 = [G:A]^2 = [G:Z]$. Then G is solvable.*

PROOF. A theorem of P. Hall (Theorem 4.5, p. 233 of [3]) asserts that a group is solvable if and only if every p-sylow subgroup has a complement. This theorem will be applied to G/A to give the result.

Since the degree of any irreducible character is bounded by the index of an abelian subgroup, A is a maximal abelian normal subgroup of G, so $Z \subseteq A$. If π, Π, and P are sylow p-subgroups of Z, A, and G respectively then $\pi \subseteq \Pi \subseteq P$. Moreover π is contained in the center of P, Π is an abelian normal subgroup of P, and Π is a normal subgroup of G.

Arguing as in [2] we show $\zeta|_P = m\lambda$ where λ is irreducible on P and $\lambda(1) = [P:\Pi]$. By Schur's lemma we have $\zeta|_Z = \zeta(1)\phi$ for some linear character ϕ of Z. Then $(\zeta, \phi^G) = (\zeta|_Z, \phi) = \zeta(1)$ so by counting degrees $\zeta(1)\zeta = \phi^G$. Let R be the subgroup of G generated by Z and P, and let λ be an irreducible character of R contained in ϕ^R. By Schur's

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lemma $\lambda|_P$ remains irreducible because the elements of Z are represented by scalars. Since λ is contained in ϕ^R, $\lambda^G=m\zeta$ for some integer m. By counting degrees

$$m = [G:R]\lambda(1)/\zeta(1).$$

Since λ is irreducible on P, $\lambda(1)=p^a$ for some a; $[G:R]$ is prime to p since R contains P; the p-part of $\zeta(1)^2$ is $[P:P\cap Z]$ and $P\cap Z=\pi$. Thus we have shown $\lambda(1)^2=[P:\pi]=(P:\Pi)^2$ and $\zeta|_P=m\lambda$. Now by Cliffords theory (p. 343 of [1])

$$\lambda|_\Pi = \alpha^1+\alpha^2+\cdots+\alpha^n$$

where the α^i are conjugate linear characters on Π (conjugate by elements in P). Let $\alpha=\alpha^1$ and $g\in G$, it follows that $\alpha^{(g)}$ is contained in $\lambda^{(g)}|_\Pi$ which in turn is contained in $\zeta^{(g)}|_\Pi=\zeta|_\Pi=m\lambda|_\Pi$. Thus $\alpha^{(g)}=\alpha^i$ for some i, and every G-conjugate of α is a P conjugate of α. To determine n observe that the p-part of $\zeta(1)$ is $[P:\Pi]=\lambda(1)=n$ which is the p-part of $[G:A]$. Let $H^*=\{g\in G|\alpha^{(g)}=\alpha\}$, then H^* is the inertia group of α in G and $[G:H^*]$ is the number of conjugates of α (p. 346 of [1]). Thus $[G:H^*]$ is the p-part of $[G:A]$ so H^*/A is a p-complement of P/A in G/A which proves the theorem.

To show that the hypothesis on the center Z of G is necessary for the theorem let H be any group and let $J_p(H)$ be the group algebra of H over the field with p-elements (p any prime). Let $A=J_p(H)$ viewed as an additive group and let H act as a group of automorphisms of A by

$$h(ax) = ahx \qquad \text{(regular representation) } x, h\in H,\ a\in J_p.$$

Let G be the semidirect product of A by H with respect to this representation of H as automorphisms of A. Then A is an abelian normal subgroup of G. Let θ be the linear character on A defined by $\theta(ax)=\xi^a\delta_{x,1}$ where 1 is the identity in H, ξ is a primitive pth root of 1, and a is the least positive integer representing the corresponding class in J_p. It is easy to see that θ has $[H:1]$ distinct conjugates under the action of G so θ^G is irreducible and $\theta(1)=[G:A]$. Observe that $H=G/A$ is arbitrary in this construction.

It is also easy to observe that if G is a group satisfying all the hypotheses of the theorem and if we let $H=G/A$ then the natural semidirect product of A by H also satisfies the hypotheses of the theorem.

There is a central extension of $A_4\times C_3$ (A_4 the alternating group of order 12, C_3 the cyclic group of order 3) which is a group with an irreducible character of large degree, this group is not nilpotent.

Using [2] (Theorem 1) the relationship between the groups we are studying, projective representations, and the Schur multiplier can be pointed out. If α is in the Schur multiplier of the group G and K is the complex field then $(KG)_\alpha$ denotes the corresponding projective group algebra.

COROLLARY. *There is a group H and an element α in the Schur multiplier of H so that $(KH)_\alpha$ has center K and an abelian normal subgroup A of H with $[H:A]^2=[H:1]$ if and only if there is a central extension G of H satisfying the hypothesis of the theorem.*

PROOF. The corollary is immediate on combining the theorem and Theorem 1 of [2].

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COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521