ON THE STABLE HOMOTOPY OF QUATERNIONIC AND COMPLEX PROJECTIVE SPACES

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ABSTRACT. Let the image in $H_{4k}(\mathrm{QP}^\infty: Z) = Z$ of stable homotopy under the Hurewicz homomorphism be $h(k) \cdot Z$. Using the Adams spectral sequence for the 2-primary stable homotopy of quaternionic and complex projective spaces it is shown that $h(k)$ is $(2k)!$ if k is even and is $(2k)!/2$ if k is odd.

1. **Introduction.** Let QP^∞ denote infinite quaternionic projective space. We prove

THEOREM 1.1. *For $k>0$ the stable Hurewicz homomorphism*

$$\pi^S_{4k}(\mathrm{QP}^\infty) \to H_{4k}(\mathrm{QP}^\infty; Z) \cong Z$$

maps onto $(2k)! \cdot Z$ for k even, onto $[(2k)!/2] \cdot Z$ for k odd.

A related result concerns the stable homotopy of complex projective spaces:

THEOREM 1.2. *Let CP^k denote k-dimensional complex projective space ($2k$ real dimensions). Let μ_k be, as in [1], the generator of a Z_2 summand of $\pi^S_{8k+1}(S^0)$ which is represented in the Adams spectral sequence by $P^k h_1$. Let 1_2 denote the integral generator of $\pi^S_2(\mathrm{CP}^k)$. Then $\mu_n 1_2$ is nonzero in $\pi^S_{8n+3}(\mathrm{CP}^{4n+1})$ and is zero in $\pi^S_{8n+3}(\mathrm{CP}^{4n+2})$.*

2. **Some standard results.** Let y generate $H^4(\mathrm{QP}^\infty; Z)$. Let η be the standard quaternionic line bundle over QP^∞, $\eta - 2 = \mu \in KU^0(\mathrm{QP}^\infty)$. Let $h(k)$ be the positive integer such that the image of the stable Hurewicz homomorphism in $H^{4k}(\mathrm{QP}^\infty; Z) = Z$ is $h(k) \cdot Z$. Since (cf. Mosher, [2]) the image of the stable Hurewicz homomorphism in $H_{4k}(\mathrm{CP}^\infty; Z) = Z$ is $(2k)! \cdot Z$ it is clear that $h(k)$ divides $(2k)!$. On the other hand using the fact that $ch\mu = \sum_{n=1} [2y^n/n!]$ it is clear that $(2k)!/2$ divides $h(k)$. For k even this last statement can be strengthened by a factor of 2 so that $(2k)!$ divides $h(k)$; one exploits the fact

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that, for k even, the natural reduction map $KSp^0(S^{4k}) \dashrightarrow KU^0(S^{4k})$ is a doubling map. To prove Theorem 1.1 it remains only to show that for k odd $h(k)$ is $(2k)!/2$ and not $(2k)!$.

3. **Remarks on** $\mathrm{Ext}_A(\tilde{H}^*(\mathrm{CP}^n; Z_2), Z_2)$. Let $E_r(X)$ denote the rth term of the Adams spectral sequence for $\pi_*^S(X; 2)$, the 2-primary component of the stable homotopy of X. $E_r(X)$ is an $E_r(S^0)$ left module and $E_2^{s,t}(X)$ is just $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(X; Z_2), Z_2)$ where A is the mod 2 Steenrod algebra.

We filter $E_2(\mathrm{CP}^n)$, $n \leqq \infty$, by setting $F^{2k}E_2(\mathrm{CP}^n)$ to be the image of $E_2(\mathrm{CP}^k)$ for $k<n$ and $E_2(\mathrm{CP}^n)$ for $k \geqq n$; $F^{2k+1}=F^{2k}$. We may then choose generators for $E_2(\mathrm{CP}^n)$ denoted by symbols of the form ${}_{2k}g$ so that g is a generator of $E_2^{s,t-2k}(S^0)$ and ${}_{2k}g$ is of exact filtration $2k$ and comes from an element of $E_2^{s,t}(\mathrm{CP}^k)$ which under the induced map of the pinching map $\mathrm{CP}^k \to S^{2k}$ goes into $g \cdot 1_{2k}$ in $E_2^{s,t}(S^{2k})$ where 1_{2k} generates $E_2^{0,2k}(S^{2k})$.

We are interested particularly in two sorts of generators in $E_2(\mathrm{CP}^n)$:

(i) The 'Z-towers'; and

(ii) Certain elements of filtration 2 and 4 on the 'top edge'.

The 'Z-towers': A Z-tower is a family of elements $h_0^n \cdot b$ which are nonzero for all $n \geqq 0$ and such that $b \neq h_0 \cdot b'$. There is just one Z-tower in the $2k$-stem of $E_2(\mathrm{CP}^\infty)$; it consists precisely of the elements of exact filtration $2k$. All of the tower, save for a possible finite segment at the bottom, persists to $E_\infty(\mathrm{CP}^\infty)$ and represents a subgroup of the integral summand of $\pi_{2s}^S(\mathrm{CP}^\infty; 2)$. But since multiplication by 2 (or any other element of $\pi_*^S(S^0; 2)$) cannot increase exact filtration this subgroup of the integral summand turns out to be the entire summand. We may therefore say that the Z-tower in the $2k$-stem of $E_2(\mathrm{CP}^\infty)$ is generated by a bottom element of the form ${}_{2k}h_0^{d(k)}$ where $d(k) \leqq k-\alpha(k)$ and that just those elements of the tower of the form $h_0^n \cdot {}_{2k}h_0^{d(k)}$ with $n \geqq k-\alpha(k)-d(k)$ will persist to $E_\infty(\mathrm{CP}^\infty)$. Here $\alpha(k)$ stands for the sum of the digits of the dyadic expansion of k. (Conjecture: If $2^{s-1} \leqq k < 2^s$ then $d(k)=2^s-k-1$.)

Since CP^∞ is an H-space, $E_r(\mathrm{CP}^\infty)$ and $\pi_*^S(\mathrm{CP}^\infty; 2)$ have product structures; $\pi_*^S(\mathrm{CP}^\infty; 2)/\text{torsion}$ is a polynomial ring over the integers (without identity) generated by 1_2 (cf. [**2**, Theorem 2.1]). Then for m, n such that $\alpha(m+n)=\alpha(m)+\alpha(n)$ we must have

$$\left(h_0^{m-\alpha(m)-d(m)} \cdot {}_{2m}h_0^{d(m)}\right)\left(h_0^{n-\alpha(n)-d(n)} \cdot {}_{2n}h_0^{d(n)}\right) = h_0^{m+n-\alpha(m+n)-d(m+n)} \cdot {}_{2(m+n)}h_0^{d(m+n)};$$

in words, the $2m$-stem Z-tower times the $2n$-stem Z-tower gives the

$2(m+n)$ Z-tower when there is no dyadic carryover in adding m to n. The proof has been in $E_\infty(\mathrm{CP}^\infty)$ but the same is true in $E_r(\mathrm{CP}^\infty)$, $r<\infty$.

The 'top edge': The maps $S^2 \xrightarrow{i} \mathrm{CP}^2 \xrightarrow{p} S^4$ induce a short exact sequence in cohomology and so a long exact sequence

$$\to E_2^{s,t}(\mathrm{CP}^2) \xrightarrow{p*} E_2^{s,t}(S^4) \xrightarrow{\partial} E_2^{s+1,t}(S^2) \xrightarrow{i*} E_2^{s+1,t}(\mathrm{CP}^2) \to$$

where ∂ is given by $\partial(g\cdot 1_4)=h_1 g\cdot 1_2$.

We have well-defined generators

$$_2P^p h_1 \in E_2^{4p+1,12p+4}(\mathrm{CP}^2), \qquad {}_4P^{p-1}h_0^3 h_3 \in E_2^{4p,12p+3}(\mathrm{CP}^2)$$

for $p\geqq 1$; they are the two 'highest' elements (in value of s) in the $8p+3$-stem of $E_2(\mathrm{CP}^2)$ and their images in $E_2(\mathrm{CP}^k)$, $k\leqq 4p+1$, are the two highest elements in the $8p+3$ stem there.

LEMMA. *In* $E_2(\mathrm{CP}^k)$, $h_0\cdot {}_4P^{p-1}h_0^3h_3={}_2P^p h_1$, $k\leqq 4p+1$.

PROOF. Write down actual resolutions for $k=2$.

4. The main theorems.

THEOREM 4.1. *If* $2n=4$ (mod 8), $\alpha(n)=r$, $n\neq 2$, *then in* $E_r(\mathrm{CP}^\infty)$, $d_r({}_{2n}h_0^{r-1})={}_2P^{(n-2)/4}h_1$.

PROOF. By induction on r. Let $r=2$, (the minimum value) $2n=2^k+4$, $k\geqq 3$, and let us examine $E_*(\mathrm{CP}^n)$. We have a long exact sequence

$$\cdots \to E_2^{s,t}(\mathrm{CP}^n) \to E_2^{s,t}(S^{2n}) \xrightarrow{\delta} E_2^{s+1,t}(\mathrm{CP}^{n-1}) \to E_2^{s+1,t}(\mathrm{CP}^n) \to \cdots.$$

Suppose there is no nonzero ${}_{2n}h_0^{n-3}\in E_2^{n-3,3n-3}(\mathrm{CP}^n)$. This would imply $\delta(h_0^{n-3}\cdot 1_{2n})\neq 0$, and then

$$\delta(h_0^{n-3}\cdot 1_{2n}) = {}_4P^{(n-6)/4}h_0^2h_3$$

and

$$\delta(h_0^{n-2}\cdot 1_{2n}) = {}_2P^{(n-2)/2}\ h_1$$

and therefore there could be no nonzero element ${}_{2n}h_0^{n-2}\in E_2(\mathrm{CP}^n)$; this is impossible and so, in fact $0\neq {}_{2n}h_0^{n-3}\in E_2^{n-3,3^n-3}(\mathrm{CP}^n)$. But since in fact this element must be killed by the time we get to E_∞ we must have

$$d_2({}_{2n}h_0^{n-3}) = {}_2P^{(n-2)/4}h_1.$$

Now let n be given such that $2n\equiv 4 \pmod 8$, $\alpha(n)=r$, $n=n'+n''$ where n' is the highest power of 2 less than n and, accordingly, $\alpha(n'')=r-1$. In $E_{r-1}(\mathrm{CP}^\infty)$ we have that ${}_{2n'}h_0^{n'-1}\neq 0$ and, by induction, we may assume that ${}_{2n''}h_0^{n''-r}\neq 0$. Hence ${}_{2n}h_0^{n-r-1}\neq 0$. There are two possibilities for eliminating ${}_{2n}h_0^{n-r-1}$;

(i) $d_{r+1}({}_{2n}h_0^{n-r-1})={}_4P^{(h-6)/4}h_0^3h_3$,

(ii) $d_r({}_{2n}h_0^{n-r-1})={}_2P^{(n-2)/4}h_1$.

Possibility (i) would imply however that $d_{r-1}({}_{2n}h_0^{n-r})={}_2P^{(n-2)/4}h_1$ which is impossible. We have therefore that ${}_{2n}h_0^{n-r-1}$ persists to E_r where it is killed. This concludes the proof of Theorem 4.1. Theorem 1.2 is an immediate corollary.

Let us now consider $E_*(\mathrm{QP}^\infty)$ using the same sort of terminology as before. We wish to show what happens to the Z-tower in the $2n$ stem, $2n\equiv 4 \pmod 8$. Specifically we are interested in knowing whether ${}_{2n}h_0^{n-r-1}$ persists to E_∞, $r=\alpha(n)$. We know that ${}_{2n}h_0^{n-r-1}$ is nonzero in $E_r(\mathrm{CP}^\infty)$ and so also in $E_r(\mathrm{QP}^\infty)$. But in $E_r(\mathrm{QP}^\infty)$ this element is trapped, there being no element ${}_2P^{(n-2)/4}h_1$ to send ${}_{2n}h_0^{n-r-1}$ into under d_r and certainly nothing with higher s-filtration. We have proved Theorem 1.1.

References

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