A NOTE ON PALAIS' AXIOMS FOR SECTION FUNCTORS

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ABSTRACT. By a slight strengthening of one axiom, a technical slip is corrected in R. S. Palais' proof of a basic lemma on functors from vector bundles over compact manifolds to Banach spaces of sections.

For each compact n-dimensional C^∞ manifold M, possibly with boundary, let $VB(M)$ denote the category of (finite dimensional real) C^∞ vector bundles and C^∞ vector bundle maps over M, and for each ξ in some $VB(M)$, let $S(\xi)$ and $C^\infty(\xi)$ denote respectively the real vector spaces of all sections and of all C^∞ sections of ξ. In [**1**], R. S. Palais studies ways of assigning to each such ξ a Banachable space $\mathfrak{M}(\xi)$ which obeys

$$C^\infty(\xi) \subset \mathfrak{M}(\xi) \subset S(\xi)$$

and certain other requirements, of which the first two are as follows [**1**, pp. 9–10]:

AXIOM (B§1). For each M, $\mathfrak{M}$ is a functor from $VB(M)$ to the category of Banachable spaces and continuous linear maps.

AXIOM (B§2). If $\xi \in VB(N)$ and if $\phi: M \to N$ is a diffeomorphism of M into N, then $s \mapsto s \circ \phi$ defines a continuous linear map of $\mathfrak{M}(\xi)$ into $\mathfrak{M}(\phi^*\xi)$.

From these properties he immediately deduces the fundamental

"MAYER-VIETORIS" THEOREM. *Let $M_1, \cdots, M_r$ be compact C^∞ submanifolds of M whose interiors cover M, and let $\xi \in VB(M)$. Define*

$$\tilde{\mathfrak{M}}(\xi) = \left\{(s_1, \cdots, s_r) \in \bigoplus_{i=1}^{r} \mathfrak{M}(\xi \mid M_i): s_j \mid M_k = s_k \mid M_j\right\}.$$

Then the map F: $\mathfrak{M}(\xi) \to \tilde{\mathfrak{M}}(\xi)$ defined by $s \mapsto (s \mid M_1, \cdots, s \mid M_r)$ is an isomorphism of Banachable spaces.

Unfortunately this theorem is false:[1] F need not be surjective, as is shown by the example below. In the proof [**1**, pp. 10–11], Palais

Received by the editors December 24, 1969.

AMS Subject Classifications. Primary 5730, 5750.

Key Words and Phrases. Banach spaces of sections, section functors, Mayer-Vietoris theorem.

[1] Also pointed out by David Ragozin.

denotes by $\{\phi_i\}$ a partition of unity subordinate to the cover $\{\text{interior } M_i\}$. Given $(s_1, \cdots, s_r) \in \tilde{\mathfrak{M}}(\xi)$, he argues from (B§1) that $\phi_i s_i \in \mathfrak{M}(\xi | M_i)$, and claims that extending $\phi_i s_i$ to $\bar{s}_i$ on M by setting it equal to zero off M_i defines $\bar{s}_i$ in $\mathfrak{M}(\xi)$. This claim is false; there is no guarantee that $\mathfrak{M}(\xi)$ is large enough to contain $\bar{s}_i$, and the appeal [1, p. 11, top] to the previous Localization Theorem 4.1 is invalid, since its hypotheses cannot be verified in the case at hand.

However, the "Mayer-Vietoris" Theorem is true if Axiom (B§2) is strengthened to require that $s \mapsto s \circ \phi$ map $\mathfrak{M}(\xi)$ *onto* $\mathfrak{M}(\phi^*\xi)$, not just into. Indeed, given $(s_1, \cdots, s_r) \in \tilde{\mathfrak{M}}(\xi)$, first extend each s_i to $t_i \in \mathfrak{M}(\xi)$ by the "onto" assertion of the strengthened (B§2). Then set $\bar{s}_i = \phi_i t_i \in \mathfrak{M}(\xi)$, $s = \bar{s}_1 + \cdots + \bar{s}_r \in \mathfrak{M}(\xi)$, and retrace Palais' proof that $F(s) = (s_1, \cdots, s_r)$.

As Palais remarks [1, p. 10], all "natural" examples of $\mathfrak{M}$ obey the "onto" form of (B§2), so the flaw noted here in no way impairs the theory developed in [1]. In particular, one has the functors $\mathfrak{M} = C^k$ (i.e., k-times continuously differentiable sections with the usual C^k-topology).

To see that the "into" form of (B§2) is not sufficient, consider the following "unnatural" choice of $\mathfrak{M}$. If dim $M = 1$, the connected components of M are diffeomorphs of the closed interval D^1 and the circle S^1. Write M_c (respectively M_n) for the disjoint union of the contractible (noncontractible) components of M, so that M = disjoint union $M_c \cup M_n$. Now for any M, if $\xi \in VB(M)$, set

$$\begin{aligned} \mathfrak{M}(\xi) &= C^0(\xi | M_c) \oplus C^1(\xi | M_n) \quad \text{if } \dim M = 1, \\ &= C^1(\xi) \quad \text{if } \dim M \neq 1. \end{aligned}$$

Trivially $\mathfrak{M}$ obeys (B§1), and the only doubt about (B§2) is in the case when dim $M = 1$. Consider diffeomorphisms ϕ of M into N. Since components are carried into components, we need consider only connected M and N. If $M = N$ ($= D^1$ or $= S^1$), Axiom (B§2) is satisfied, as noted above, even in the "onto" form. If $M = S^1$, $N = D^1$, there are no such ϕ. If $M = D^1$, $N = S^1$, then $s \in C^1(\xi)$ "restricts" to

$$s \circ \phi \in C^1(\phi^*\xi) \subset C^0(\phi^*\xi) = \mathfrak{M}(\phi^*\xi),$$

and the map $s \mapsto s \circ \phi$ is continuous since the C^1-topology is stronger than the C^0. Note that the map $s \mapsto s \circ \phi$ is decidedly not onto $\mathfrak{M}(\phi^*\xi)$, so this $\mathfrak{M}$ obeys only the "into" form of (B§2). And now the "Mayer-Vietoris" Theorem obviously fails, e.g. for the case $M = S^1$, M_1 and M_2 = submanifolds diffeomorphic to D^1.

Finally, we observe that the local equivalent of (B§2), namely Axiom (B§2′) on p. 12, should also be changed to the stronger "onto" form.

References

1. Richard S. Palais, *Foundations of global non-linear analysis*, Benjamin, New York, 1968.

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