ON THE EXISTENCE OF $L_{\infty\kappa}$-INDISCERNIBLES

P. C. EKLOF

ABSTRACT. It is proved that if T is a countable theory of $L_{\omega_1\omega}$ with enough axioms for Skolem functions and with arbitrarily large models, then for any order type, there is a model of T with a set of $L_{\infty\kappa}$-indiscernibles of that order type.

In this short note we answer in the affirmative a question of Chang [4] as to whether there exist $L_{\kappa\kappa}$-indiscernibles of any given order type. In fact we prove a somewhat stronger result since we show the existence of $L_{\infty\kappa}$-indiscernibles and we use a stronger definition of indiscernibles. Our result also gives a simpler proof of Chang's theorem on the existence of $L_{\kappa\kappa}$-indiscernibles of any well-ordered order type [4, Theorem 4]. (We thank Jon Barwise for some helpful discussions.)

In general we follow the notation of Chang [4] (so κ is always an infinite regular cardinal). Let L be a first order language with countably many relation, function, and constant symbols, and let $\mathfrak{A}=\langle A, \cdots\rangle$ be a structure for L. An ordered subset X of A is said to be $L_{\lambda\kappa}$-*indiscernible* if for any subset Y of X of cardinality $<\kappa$ and any order-preserving injection $h: Y\rightarrow X$,

$$(\mathfrak{A}, y)_{y\in Y} \equiv_{\lambda\kappa} (\mathfrak{A}, hy)_{y\in Y}.$$

If $L_{\infty\kappa}$ is the union of the infinitary languages $L_{\lambda\kappa}$, where λ ranges over all cardinals, we define $L_{\infty\kappa}$-indiscernibles analogously.

Let T be a countable theory of $L_{\omega_1\omega}$. There is a countable fragment $\mathcal{L}_A$ such that $T\subseteq\mathcal{L}_A$ (for the definition of $\mathcal{L}_A$ see Barwise [1]). We consider only T and $\mathcal{L}_A$ such that $\mathcal{L}_A$ has enough function symbols and T includes axioms for all Skolem functions of formulas of $\mathcal{L}_A$. A necessary condition for T to have models with sets of $L_{\infty\kappa}$-indiscernibles of any order type is that T have models of arbitrarily large cardinality; this is sufficient as well.

THEOREM. *Let $T\subseteq\mathcal{L}_A$ such that T has arbitrarily large models. If μ is any order type, there is a model $\mathfrak{A}$ of T such that $\mathfrak{A}$ has a set of $L_{\infty\kappa}$-indiscernibles of order type μ.*

PROOF. We may suppose that the cardinality $|\mu|$ of μ is $\leqq\kappa$, since a set of $L_{\infty\lambda}$-indiscernibles is a set of $L_{\infty\kappa}$-indiscernibles if $\lambda\geqq\kappa$. Sup-

Received by the editors December 7, 1969.

AMS Subject Classifications. Primary 0235.

Key Words and Phrases. Indiscernibles, infinitary languages, η_α-set.

pose $\kappa = \aleph_\alpha$; it suffices to prove that T has a model $\mathfrak{A}$ with a set X of $L_{\infty\kappa}$-indiscernibles of order type η_α, since μ can be embedded in X [7, pp. 334–338].

We are assuming that models of T have Skolem functions for all formulas of $\mathfrak{L}_A$. Since T has arbitrarily large models, there is a model $\mathfrak{A}$ of T with a set X of $\mathfrak{L}_A$-indiscernibles of order type η_α (see [6]; if $\mathfrak{L}_A = L_{\omega\omega}$ this is just the classical result of Ehrenfeucht-Mostowski [5]). We may suppose that $\mathfrak{A} = \mathfrak{H}(X)$, where $\mathfrak{H}(X)$ is the Skolem hull of X (i.e. the submodel of $\mathfrak{A}$ whose universe A is the closure of X under the Skolem functions of $\mathfrak{L}_A$).

We claim that X is a set of $L_{\infty\kappa}$-indiscernibles in $\mathfrak{A}$. Let $Y \subseteq X$ be of cardinality $< \kappa = \aleph_\alpha$ and let $h: Y \to X$ be an order-preserving injection. Let I be the set of all isomorphisms

$$f: S \to S'$$

of submodels S, S' of $\mathfrak{A}$ such that $Y \subseteq S$, $f | Y = h$, and there exist U, $U' \subseteq X$ such that $|U| < \kappa$, $S = \mathfrak{H}(U)$, $S' = \mathfrak{H}(U')$ and $f | U$ is an order-isomorphism of U onto U'. Notice that $I \neq \emptyset$ since, letting $S = \mathfrak{H}(Y)$, $S' = \mathfrak{H}(h(Y))$, there is an extension of h to an isomorphism $f: S \to S'$. We claim that I has the following property:

(*) For any $C \subseteq A$ such that $|C| < \kappa$ and any $f \in I$, there are f', $f'' \in I$ such that $f \subseteq f'$, $f \subseteq f''$, $C \subseteq$ domain of f', and $C \subseteq$ range of f''.

It suffices to prove (*), for it follows easily by an induction on formulas of $L_{\infty\kappa}$ that

$$(\mathfrak{A}, y)_{y \in Y} \equiv_{\infty\kappa} (\mathfrak{A}, hy)_{y \in Y}$$

(see Calais [2]).

So suppose $f: S \to S'$ is in I and U, U' are as in the definition of I. Given $C \subseteq A$, $|C| < \kappa$, there is a $D \subseteq X$, $|D| < \kappa$, such that $C \subseteq \mathfrak{H}(U \cup D)$. It is clear that in order to prove the existence of f' as required by (*), it suffices to show that we can extend $f | U: U \to U'$ to an order-monomorphism: $U \cup D \to X$. We may assume $D \cap U = \emptyset$. Define an equivalence relation on D by: $x \approx y$ iff x and y determine the same cut of U. Write $D = \bigcup_{\sigma < \tau} D_\sigma$ as the union of pairwise disjoint equivalence classes D_σ, $\sigma < \tau < \kappa$. For any $\sigma < \tau$, let $U = A_\sigma \cup B_\sigma$ where $A_\sigma < D_\sigma < B_\sigma$. Then $f(A_\sigma) < f(B_\sigma)$ and $|f(A_\sigma)| < \kappa$, $|f(B_\sigma)| < \kappa$. So if

$$E_\sigma = \{x \in X : f(A_\sigma) < x < f(B_\sigma)\},$$

E_σ is an η_α-set. Therefore there exists an embedding

$$g_\sigma : D_\sigma \to E_\sigma.$$

Define $f' : U \cup D \to X$ by: $f'(x) = f(x)$ if $x \in U$; $f'(x) = g_\sigma(x)$ if $x \in D_\sigma$. This gives the desired extension of f. In a similar manner we can prove the existence of f'' extending f with $C \subseteq$ range of f''. This completes the proof.

REMARKS. (1) If we assume the generalized continuum hypothesis then the proof is much simpler; for then there exists an η_α-set X of cardinality $\aleph_\alpha$. Hence if $h: Y \to X$ is an order-preserving injection and $|Y| < \kappa$, h extends to an isomorphism $h' : X \to X$. It is immediate that

$$(\mathfrak{A}, y)_{y \in Y} \equiv_{\infty\kappa} (\mathfrak{A}, hy)_{y \in Y}.$$

(Compare the remark of Chang [**3**, p. 55].)

(2) Our method suffers from the same defect as that of Chang, namely the indiscernibles do not necessarily generate the model.

(3) If $\kappa = \aleph_\alpha$ and $\kappa \geqq |\mu|$ the model $\mathfrak{A}$ asserted to exist in the statement of the theorem can be chosen to have cardinality $= 2^{\aleph_\beta}$ if $\alpha = \beta + 1$; $\sum_{\sigma < \alpha} 2^{\aleph_\sigma}$ if α is a limit ordinal [**7**].

REFERENCES

1. J. Barwise, *Infinitary logic and admissible sets*, J. Symbolic Logic **34** (1969), 226–252.

2. J.-P. Calais, *La méthode de Fraissé dan les langages infinis*, C. R. Acad. Sci. Paris **268** (1969), 785–788.

3. C. C. Chang, "Some remarks on the model theory of infinitary languages," *The syntax and semantics of infinitary languages*, Lecture Notes in Math., no. 72, Springer-Verlag, Berlin and New York, 1968, pp. 36–63.

4. ———, "Infinitary properties of models generated from indiscernibles," *Logic, methodology and the philosophy of science*. III, North-Holland, Amsterdam, 1968, pp. 9–21.

5. A. Ehrenfeucht and A. Mostowski, *Models of axiomatic theories admitting automorphisms*, Fund. Math. **43** (1956), 50–68. MR **18**, 863.

6. J. Keisler, *Model theory of* $L_{\omega_1\omega}$, (to appear).

7. K. Kuratowski and A. Mostowski, *Set theory*, PWN, Warsaw and North-Holland, Amsterdam, 1968. MR **37** #5100.

YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520