CONULLITY OF OPERATORS ON SOME *FK*-SPACES

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ABSTRACT. The notion of *conullity* for a subclass of the algebra of matrix operators on the space of convergent sequences is well known in summability theory. In this paper the space of convergent sequences is replaced by a general (locally convex) *FK*-space and the following question is studied: Given a subalgebra of the algebra of all continuous linear operators on this *FK*-space, is there a class of operators in this subalgebra whose behavior is "conull-like"? The question is answered in the case when the *FK*-space has a suitable (Schauder) basis and also in some other special cases.

1. **Introduction.** The algebra, Γ_c, of complex matrix operators on the set c of convergent complex number sequences is partitioned into two classes: the conull matrices and the coregular matrices. (See, for example, [1].) The class of conull matrices, denoted by ψ, may be characterized either as the kernel of the only nontrivial multiplicative linear functional on Γ_c ([1]; see also [6]), or as the set $\{A \in \Gamma_c: w^r \to 0$ weakly in $c_A\}$, where c_A denotes the summability field of A and $w^r = e - \sum_{k=1}^{r} e^k$, for $r = 1, 2, \cdots$. (As usual, $e = (1, 1, 1, \cdots)$ is the unit sequence and $e^k = (0, \cdots, 0, 1, 0, \cdots)$ is the sequence having a 1 in the kth coordinate and zeros elsewhere.) Since the kernel of a multiplicative functional is an ideal, the first characterization displays an algebraic likeness between the set of conull matrices and the zero matrix (in the sense that $A\psi \subseteq \psi$ and $\psi A \subseteq \psi$ for each A in Γ_c, just as $A\{0\} \subseteq \{0\}$ and $\{0\}A \subseteq \{0\}$, where 0 denotes the zero matrix). The second characterization displays a topological likeness between ψ and the zero matrix (because $w^r \to 0$ weakly in s, the set of all complex number sequences, and s is the summability field of the zero matrix). In this paper we replace c by a more general (locally convex) *FK*-space λ and we replace Γ_c by the set Γ_λ of matrix operators on λ. We then consider the following question: Given an algebra Λ which contains Γ_λ and which is contained in the set $B[\lambda]$ of all continuous linear operators on λ, is there a class of operators in Λ whose behavior is "conull-like"? By "conull-like" we will mean that the class resembles the zero matrix in both an algebraic and a topo-

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logical sense. In order to make this more precise we first give an equivalent reformulation of the second characterization of ψ.

Recall that if $A\in\Gamma_c$ and $f\in c'_A$, the dual space of c_A, then f has representation[3] [4, p. 230]

$$f(x) = g(Ax) + \sum_k \beta_k x_k,$$

where g is a continuous linear functional on c and $\sum_k |\beta_k| < \infty$. But

$$\lim_r \sum_k \beta_k w_k^r = \lim_r \sum_{k=r+1}^{\infty} \beta_k = 0$$

and so $w^r \to 0$ weakly in c_A if and only if $Aw^r \to 0$ weakly in c. Thus, ψ may be characterized as being precisely the set

$$\{A \in \Gamma_c : Aw^r \to 0 \text{ weakly in } c, \text{ as } r \to \infty\}.$$

(Notice that this equivalent reformulation also displays a topological likeness between ψ and the zero matrix in the sense that it shows that each A in ψ maps a sequence, each of whose elements is at a distance of one from the zero sequence, into a weakly convergent to zero sequence.)

Now let λ be an FK-space, $\{x^r : r=1, 2, \cdots\}$ a subset of λ, and Λ a subalgebra of $B[\lambda]$ which contains Γ_λ. Furthermore, let

$$(\Lambda, x^r, \lambda) = \{A \in \Lambda : Ax^r \to 0 \text{ weakly in } \lambda, \text{ as } r \to \infty\}.$$

We say that "$\{x^r\}$ in λ acts like $\{w^r\}$ in the sense of Wilansky," and write $x^r \sim \lambda$, if $x^r \to 0$ in s but $x^r \not\to 0$ weakly in λ. (See [5, p. 90].) We can now make precise what we mean by "conull-like" by reformulating our original question as follows: given λ and Λ, does there exist a subset $\{x^r\}$ of λ such that $x^r \sim \lambda$ and such that (Λ, x^r, λ) is a proper ideal in Λ? If the answer is affirmative, that is, if $x^r \sim \lambda$ and if (Λ, x^r, λ) is a proper ideal in Λ, then we say that λ *is* Λ-*conullable in* Λ *under* $\{x^r\}$. For example, c is Γ_c-conullable in Γ_c under $\{w^r\}$ because $w^r \sim c$ and $(\Gamma_c, w^r, c) = \psi$ is a proper ideal in Γ_c.

The zero matrix satisfies a stronger property; namely that $0x^r \to 0$ in the topology of λ. This motivates the following definition. If $x^r \sim \lambda$ and if the set

$$(\Lambda^*, x^r, \lambda) = \{A \in \Lambda : Ax^r \to 0 \text{ in } \lambda, \text{ as } r \to \infty\}$$

is a proper ideal in Λ, then we say that λ *is* Λ^*-*conullable in* Λ *under* $\{x^r\}$.

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[3] Unless otherwise specified, all summations are from 1 to ∞.

It is clear that for each λ, Λ, and x^r in λ, $(\Lambda^*, x^r, \lambda)$ is a subset of (Λ, x^r, λ).

In the course of our work we establish the following results. (All as yet undefined symbols will be defined in §2.)

(i) If $\{e^r\}$ is a (Schauder) basis for λ (for example, if λ is l^p $(p \geqq 1)$, c_0, or γ), then λ is not Γ_λ-conullable in Γ_λ $(=B[\lambda])$ under $\{e^r\}$.

(ii) If $\{u, e^k : k=1, 2, \cdots\}$ is a (Schauder) basis for λ (for example, if $u=e$ and λ is either c or v), then (Λ, z^r, λ) is always a class of matrices, where $z^r = u - \sum_{k=1}^{r} u_k e^k$. Furthermore, λ is Λ-conullable in Λ under $\{z^r\}$ if and only if $\Lambda = \Gamma_\lambda$. In other words, Λ-conullity of λ under $\{z^r\}$ is, and only is, a matrix notion whenever $\{u, e^k\}$ is a basis for λ.

(iii) If λ is either c or v, and if λ is Λ-conullable under $\{x^r\}$ in Λ, then $\Lambda = \Gamma_\lambda$ and $(\Lambda, x^r, \lambda) = (\Lambda, w^r, \lambda)$. That is, Λ-conullity in either c or v is a unique matrix concept.

(iv) m is not Λ-conullable in any Λ under any $\{x^r\}$.

(v) If λ is l^p $(p \geqq 1)$, c_0, γ, c, or m, then λ is not Λ^*-conullable in any Λ under any $\{x^r\}$. However, if λ is v then the only Λ and $\{x^r\}$ which make v Λ^*-conullable in Λ under $\{x^r\}$ are Γ_v and $\{w^r\}$; moreover, $(\Gamma_v^*, w^r, v) = (\Gamma_v, w^r, v) = \theta$, where θ denotes the set of compact matrices on v.

Since the set of compact operators in $B[\lambda]$ is always an ideal and since a compact operator resembles the zero matrix in how it maps sets, an argument could be made for selecting the set of compact operators as representing the "conull-like" class of operators. However, from a summability point of view this is not satisfactory because conullity is an invariant property, i.e. if A is conull and $\lambda_A = \lambda_B$ then B is also conull (the case when $\lambda = c$ is very well known), whereas compactness is not an invariant property. For example, take λ to be γ (the set of convergent series) and take A to be the matrix whose first row is e and whose other entries are zero. Then A is a compact operator on γ and γ_A $(=\{x : Ax \in \gamma\})$ is precisely γ. But γ is also γ_I, where I is the identity matrix which, of course, is not compact.

2. **Further definitions and notation.** As usual, l^p $(p \geqq 1)$, m, c, c_0, γ, and v, respectively, denote the subsets of s consisting of those sequences x for which $\sum_k |x_k|^p < \infty$, $\sup_k |x_k| < \infty$, $\lim_k x_k$ exists, $\lim_k x_k = 0$, $\sum_k x_k$ converges, and $\sum_k |x_k - x_{k+1}| < \infty$, respectively. (In the particular case when $p=1$ we will write l instead of l^1.) The β-dual of an *FK*-space λ, denoted by λ^β, is the set

$$\lambda^\beta = \left\{\beta \in s;\ \sum_k \beta_k x_k \text{ converges for all } x \in \lambda\right\}.$$

By the Banach-Steinhaus Closure Theorem, $\sum_k \beta_k x_k$ defines a member of the dual space λ' of λ for each $\beta \in \lambda^\beta$. We assume, throughout this paper, that all FK-spaces contain all the finite sequences and that the composition of any two members of Γ_λ is given by matrix multiplication. (This is the case if either $\{e^k\}$ is a Schauder basis for λ, or if Γ_λ is a ring under matrix multiplication and $\{e^k\}$ is a Schauder basis for λ^β.)

All increasing subsequences of positive integers are denoted by $\{n_k\}$ or, for convenience, by $\{n(k)\}$. The letter V is reserved for the matrix (v_{nk}) defined by the set of equations

$$v_{nk} = 1/n \quad \text{for } (n-1)n/2 < k \leqq n(n+1)/2.$$

(Throughout this paper all undesignated entries in matrices and sequences are assumed to be zero.)

Let λ be an FK-space. We say that λ is:

averaging if, given $x \in \lambda$ then $y \in \lambda$ whenever $y_k = x_k/n$ for $(n-1)n/2 < k \leqq n(n+1)/2$;

contractive if, given $\{k_i\}$ and $x \in \lambda$ then $y \in \lambda$ whenever $y_i = x_{k(i)}$;

repeating if, given $\{k_i\}$ and $x \in \lambda$ then $y \in \lambda$ whenever $y_k = x_i$ for $k_i \leqq k < k_{i+1}$;

expansive if, given $\{k_i\}$ and $x \in \lambda$ then $y \in \lambda$ whenever $y_{k(i)} = x_i$.

We remark here that each of the special spaces mentioned earlier is at least averaging and contractive.

A matrix $A = (a_{nk})$ is called:

contractive if, given $\{k_n\}$ then $a_{n,k(n)} = 1$;

repeating if, given $\{n_i\}$ then $a_{ni} = 1$ for $n_i \leqq n < n_{i+1}$;

expansive if, given $\{n_i\}$ then $a_{n(i),i} = 1$.

Thus, an FK-space λ is averaging if and only if $V \in \Gamma_\lambda$; it is contractive (resp., repeating or expansive) if and only if Γ_λ contains all contractive (resp., repeating or expansive) matrices.

3. **General results.** As mentioned above λ always represents an FK-space containing the finite sequences, Λ represents a subalgebra of $B[\lambda]$ which contains Γ_λ, and composition in Γ_λ is matrix multiplication.

LEMMA 1. *(Λ, x^r, λ) and $(\Lambda^*, x^r, \lambda)$ are left ideals in Λ.*

PROOF. Let $A \in (\Lambda, x^r, \lambda)$ and $B \in \Lambda$. Then $(BA)x^r = B(Ax^r)$ and, since B is continuous, $B(Ax^r) \to 0$ weakly in λ. Thus, $BA \in (\Lambda, x^r, \lambda)$. Similarly, $(\Lambda^*, x^r, \lambda)$ is a left ideal in Λ.

LEMMA 2. *If $x^r \sim \lambda$ and if (Λ, x^r, λ) is a right ideal in Λ, then $\sum_k \beta_k x_k^r \to 0$ for each $\beta \in \lambda^\beta$. A similar result holds if $(\Lambda^*, x^r, \lambda)$ is a right ideal in Λ.*

PROOF. Let $E=(e_{nk})$ be the matrix defined by setting $e_{11}=1$ and $e_{nk}=0$ if either $n\neq 1$ or $k\neq 1$. Since $x^r\sim\lambda$, $E\in(\Lambda^*, x^r, \lambda)\subseteq(\Lambda, x^r, \lambda)$. Given any $\beta\in\lambda^\beta$, define $B=(b_{nk})$ by setting $b_{1k}=\beta_k$. Then $B\in\Gamma_\lambda$ and $EB=B$. Hence, if $(\Lambda^*, x^r, \lambda)$ (resp., (Λ, x^r, λ)) is an ideal, then $B\in(\Lambda^*, x^r, \lambda)$ (resp., $B\in(\Lambda, x^r, \lambda)$). Moreover, $Bx^r=e^1\sum_k \beta_k x_k^r$ and so $f(Bx^r)=f(e^1)\sum_k \beta_k x_k^r\to 0$ for each $f\in\lambda'$. Since we may choose f so that $f(e^1)\neq 0$ the proof of the lemma is complete.

COROLLARY. *If $\lambda'=\lambda^\beta$ then λ is not Λ-conullable in any Λ under any $\{x^r\}$.*

Since $\lambda'=\lambda^\beta$ whenever $\{e^k\}$ is a basis for λ, we see that, in particular, l^p $(p\geqq 1)$, c_0, and γ are never Λ-conullable in Λ under any $\{x^r\}$.

LEMMA 3. *Let $\{u, e^k: k=1, 2, \cdots\}$ be a basis for λ and let $z^r=u-\sum_{k=1}^r u_k e^k$. Then $\sum_k \beta_k z_k^r\to 0$ for each $\beta\in\lambda^\beta$.*

PROOF. If $\beta\in\lambda^\beta$ then $\sum_{k=r+1}^\infty \beta_k x_k\to 0$, as $r\to\infty$, for each $x\in\lambda$. Thus, $\sum_k \beta_k z_k^r=\sum_{k=r+1}^\infty \beta_k u_k\to 0$, as $r\to\infty$.

LEMMA 4. *Let $\{u, e^k\}$ be a basis for λ and let $T\in B[\lambda]$. Then $T\in\Gamma_\lambda$ if and only if $\sum_k u_k(Te^k)n$ converges and equals $(Tu)_n$ for each n.*

(The special case when $\lambda=c$ and $u=e$ was proved by Wilansky [**6**]. Our proof is essentially his and we present it here for the sake of completeness.)

PROOF. If $x\in\lambda$ then x has the representation $x=\alpha_0 u+\sum_k \alpha_k e^k$ for some unique sequence of scalars $\alpha_0, \alpha_1, \alpha_2, \cdots$. Since coordinates are continuous in λ the representation becomes $x=\alpha_0 u+\sum_k (x_k-\alpha_0 u_k)e^k$. Thus, for each $T\in B[\lambda]$, $Tx=\alpha_0 Tu+\sum_k (x_k-\alpha_0 u_k)Te^k$ and the proof follows easily from this equation.

THEOREM 1. *Let $\{u, e^k\}$ be a basis for λ and let $z^r=u-\sum_{k=1}^r u_k e^k$.*
(i) *(Λ, z^r, λ), and hence $(\Lambda^*, z^r, \lambda)$, is contained in Γ_λ for each Λ.*
(ii) *λ is Γ_λ-conullable in Γ_λ under $\{z^r\}$.*
(iii) *If $\Lambda\neq\Gamma_\lambda$ then λ is neither Λ-conullable nor Λ^*-conullable in Λ under $\{z^r\}$.*

PROOF. (i) This is an immediate consequence of Lemma 4 because if $T\in(\Lambda, z^r, \lambda)$ then $(Tz^r)_n\to 0$, as $r\to\infty$, for each n.

(ii) Notice first that $z^r\sim\lambda$ because there exists $f\in\lambda'$ satisfying $f(u)=1$ and $f(e^k)=0$ for each k. Also, if I denotes the identity matrix then $I\notin(\Gamma_\lambda, z^r, \lambda)$ and so $(\Gamma_\lambda, z^r, \lambda)\neq\Gamma_\lambda$.

To show that λ is Γ_λ-conullable in Γ_λ under $\{z^r\}$ it suffices, by Lemma 1, to show that $(\Gamma_\lambda, z^r, \lambda)$ is a right ideal. Thus, let $B\in(\Gamma_\lambda, z^r, \lambda)$, $f\in\lambda'$, and $A\in\Gamma_\lambda$. Since $Bz^r\to 0$ weakly in λ, $\sum_k u_k f(Be^k)$ converges and equals $f(Bu)$. Hence, the sequence $\{f(Be^k)\}$ is a member of λ^β. (Indeed, if $x \in \lambda$ then there exists a scalar α_0 such that $x = \alpha_0 u + \sum_k (x_k - \alpha_0 u_k)e^k$ and so $f(Bx) = \alpha_0 f(Bu) + \sum_k (x_k - \alpha_0 u_k) f(Be^k) = \sum_k x_k f(Be^k)$.) Thus, if we define $F = (f_{nk})$ by setting $f_{1k} = f(Be^k)$, then $F \in \Gamma_\lambda$ and $FA \in \Gamma_\lambda$. Let β denote the first row of FA. Then $\beta \in \lambda^\beta$ and, by Lemma 3, $\sum_k \beta_k z_k^r \to 0$, as $r \to \infty$. But $\sum_k \beta_k z_k^r = ((FA)z^r)_1$ and since matrix multiplication is composition, $((FA)z^r)_1 = (F(Az^r))_1 = \sum_k f(Be^k)(Az^r)_k = f(B(Az^r)) = f((BA)z^r)$, so that $(BA)z^r \to 0$ weakly in λ. Hence, $BA \in (\Lambda, z^r, \lambda)$ and so λ is Γ_λ-conullable in Γ_λ under $\{z^r\}$.

(iii) Without loss of generality we may assume that $u_j \neq 0$ for each j. Now, for each j, define $Q^j = (q_{nk}^j)$ by setting $q_{nj}^j = u_n/u_j$ and $q_{nk}^j = 0$ when $k \neq j$. Then for each $x \in \lambda$ we have that $Q^j x = (x_j/u_j)u \in \lambda$ and so $Q^j \in \Gamma_\lambda$. Moreover, $Q^j u = u$, $Q^j e^j = (1/u_j)u$, and $Q^j e^k = \{0\}$ if $k \neq j$. Therefore, $Q^j \in (\Lambda, z^r, \lambda)$.

Assume that λ is Λ-conullable in Λ under $\{z^r\}$. Then (Λ, z^r, λ) is a right ideal in Λ and so $Q^j T(u - \sum_{k=1}^r u_k e^k) \to 0$ weakly in λ, since $Q^j T \in (\Lambda, z^r, \lambda)$ for each $T \in \Lambda$. But $Q^j(Tu) = ((Tu)_j/u_j)u$, $Q^j(Te^k) = ((Te^k)_j/u_j)u$ and coordinates are continuous in λ; hence, we have that $(Tu)_j = \sum_k u_k (Te^k)_j$. Since this holds for each j, Lemma 4 shows that $T \in \Gamma_\lambda$. This contradicts the hypothesis that $\Gamma_\lambda \neq \Lambda$ and so λ is not Λ-conullable in Λ under $\{z^r\}$. A similar argument may be used to show that λ is not Λ^*-conullable in Λ under $\{z^r\}$.

THEOREM 2. *If λ is Λ-conullable (resp., Λ^*-conullable) in Λ under $\{x^r\}$ then, for each $A \in \Lambda$, $(\Lambda, x^r, \lambda) \subseteq (\Lambda, Ax^r, \lambda)$ (resp., $(\Lambda^*, x^r, \lambda) \subseteq (\Lambda^*, Ax^r, \lambda)$). If, in addition, A has a right inverse in Λ then $(\Lambda, x^r, \lambda) = (\Lambda, Ax^r, \lambda)$ (resp., $(\Lambda^*, x^r, \lambda) = (\Lambda^*, Ax^r, \lambda)$).*

PROOF. For each $B \in (\Lambda, x^r, \lambda)$ and each $f \in \lambda'$ we have that $f(B(Ax^r)) = f(BA(x^r))$. Since (Λ, x^r, λ) is a right ideal, $BA \in (\Lambda, x^r, \lambda)$; hence, $B(Ax^r) \to 0$ weakly in λ and so $B \in (\Lambda, Ax^r, \lambda)$.

Assume next that A' is a right inverse for A in Λ and let $B \in (\Lambda, Ax^r, \lambda)$. Then $BA(x^r) = B(Ax^r) \to 0$ weakly in λ and so $BA \in (\Lambda, x^r, \lambda)$. Since (Λ, x^r, λ) is a right ideal, $B = (BA)A' \in (\Lambda, x^r, \lambda)$. The parenthetical statements are proved analogously.

LEMMA 5. *Given $x^r \in m$, $r = 1, 2, \cdots$, there exists a contractive matrix A such that $Ax^r \in c$, $r = 1, 2, \cdots$.*

PROOF. Since each $x^r \in m$ we may choose $\{k_j\}$ (by a diagonalization process) so that $\lim_j x_{k(j)}^r$ exists for each r. If $A = (a_{nk})$ is defined by setting $a_{n,k(n)} = 1$, then A is contractive and $Ax^r \in c$ for each r.

LEMMA 6. *For each $r = 1, 2, \cdots$, let $y^r \in c$ with $\lim_k y_k^r = \alpha_r$. If $y^r \to 0$ in s and if $\{r_j\}$ is given then there exist a contractive matrix B, a subse-*

quence $\{r_j'\}$ *of* $\{r_j\}$, *and a sequence* $\{v^r\}$ *of elements in the unit ball of* l *with* $v^{r'(j)}\to 0$ *in* l, *as* $j\to\infty$, *such that*

$$By^r = \alpha_r w^j + v^r \quad \text{for } r_j' \leqq r < r_{j+1}'.$$

(Recall that $w^j = e - \sum_{k=1}^{j} e^k$.)

PROOF. Choose k_1 so that $|y_k^1 - \alpha_1| < 1/2$ for $k \geqq k_1$ and then choose r_1' so that $|y_{k(1)}^r| < 1/2$ whenever $r \geqq r_1'$. Having chosen k_j and r_j', choose $k_{j+1} > k_j$ so that $|y_k^r - \alpha_k| < 1/2^{j+1}$ for $k \geqq k_{j+1}$ and $1 \leqq r \leqq r_j'$ and then choose $r_{j+1}' > r_j'$ so that $|y_{k(i)}^r| < 1/2^{j+1}$ whenever $r \geqq r_{j+1}'$ and $i = 1, 2, \cdots, j+1$. Then the matrix $B = (b_{nk})$, defined by setting $b_{j,k(j)} = 1$ $(j = 1, 2, \cdots)$, is contractive and satisfies the conclusion of the lemma.

LEMMA 7. *Let* $\{v^r\}$ *be a sequence of elements in the unit ball of* l *such that* $v^r \to 0$ *in* s. *Then there exist* $\{r_j\}$, *a contractive matrix* B, *and a sequence of scalars* $\{c_r\}$ *with* $\sup|c_r| \leqq 1$, *such that*

$$Bv^r = c_r e^j + y^r \quad \text{for } r_j \leqq r < r_{j+1},$$

where each $y^r \in l$ *and* $y^r \to 0$ *in* l, *as* $r \to \infty$.

PROOF. Choose k_1 so that $\sum_{k=k(1)}^{\infty} |v_k^1| < 1/2$ and then choose r_1 so that $|v_{k(1)}^r| < 1/2$ whenever $r \geqq r_1$. Having chosen k_j and r_j, choose $k_{j+1} > k_j$ so that $\sum_{k=k(j+1)}^{\infty} |v_k^r| < 1/2^{j+1}$ for $r \leqq r_j$ and then choose $r_{j+1} > r_j$ so that $|x_{k(i)}^r| < 1/2^{j+1}$ whenever $r \geqq r_{j+1}$ and $i = 1, 2, \cdots, j+1$. Then the matrix $B = (b_{nk})$, defined by setting $b_{j,k(j)} = 1$ $(j = 1, 2, \cdots)$, is contractive and satisfies the conclusion of the lemma.

LEMMA 8. *Given* $\{n_k\}$ *there exists a contractive matrix* A *such that* $Ae^{n(k)} = e^k$ *for each* k.

The proof of this lemma is straightforward and so we omit it.

THEOREM 3. *Let* $\lambda \supseteq l$ *and suppose that whenever* $\{w^r\}$ *is a subset of* λ *then it is a bounded subset of* λ. *Let* $x^r \in m$ $(r = 1, 2, \cdots)$ *and assume also that* λ *is contractive and is such that each contractive matrix has a right inverse in* Λ. *Then, if* λ *is* Λ*-conullable* (*resp.*, Λ^**-conullable*) *in* Λ *under* $\{x^r\}$, *either*

$$(\Lambda, x^r, \lambda) = (\Lambda, e^r, \lambda) \quad (\textit{resp.}, (\Lambda^*, x^r, \lambda) = (\Lambda^*, e^r, \lambda))$$

or

$$(\Lambda, x^r, \lambda) = (\Lambda, w^r, \lambda) \quad (\textit{resp.}, (\Lambda^*, x^r, \lambda) = (\Lambda^*, w^r, \lambda)).$$

PROOF. We shall only prove the theorem for the case when λ is Λ-conullable. The proof for the other case is analogous.

By Lemma 5 there exists a contractive matrix A such that $y^r = Ax^r \in c$ for each r, and, by Theorem 2, since A has a right inverse in Λ, $(\Lambda, x^r, \lambda) = (\Lambda, y^r, \lambda)$. Notice that $y^r \to 0$ in s because $x^r \to 0$ in s. If $\alpha_r = \lim_k y_k^r$ then there are two cases to consider. Either there exists $\{r_j\}$ such that $\alpha_{r(j)} \to \alpha \neq 0$, as $j \to \infty$, or else $\alpha_r \to 0$, as $r \to \infty$.

In the first case, by Lemma 6, there exists a contractive matrix B, a subsequence $\{r_j'\}$ of $\{r_j\}$ and sequences v^r in the unit ball of l with $v^{r'(j)} \to 0$, as $j \to \infty$, such that $By^r = \alpha_r w^j + v^r$ for $r_j' \leqq r < r_{j+1}'$. Hence, using Theorem 2 (since B has a right inverse in Λ) we get

$$(\Lambda, x^r, \lambda) = (\Lambda, y^r, \lambda) = (\Lambda, \alpha_r w^j + v^r, \lambda),$$

where $r_j' \leqq r < r_{j+1}'$. But $v^{r'(j)} \to 0$ in λ (because $v^{r'(j)} \to 0$ in l and $\lambda \supseteq l$ [**4**, p. 203, Corollary 1]) and $\alpha_{r(j)} \to 0$, as $j \to \infty$; hence, $(\Lambda, x^r, \lambda) = (\Lambda, w^r, \lambda)$.

In the second case we have that $\alpha_r \to 0$, as $r \to \infty$. Using Lemma 6 and Theorem 2 as in the first case (this time with I^+ the given subsequence) we again obtain a contractive matrix B and sequences v^r such that

$$(\Lambda, x^r, \lambda) = (\Lambda, y^r, \lambda) = (\Lambda, By^r, \lambda) = (\Lambda, \alpha_r w^j + v^r, \lambda).$$

But, by hypothesis, $\{w^r\}$ is a bounded subset of λ (whenever it is a subset of λ) and so $\alpha_r w^j \to 0$ in λ, as $r \to \infty$. Hence, $(\Lambda, x^r, \lambda) = (\Lambda, v^r, \lambda)$. By applying Lemma 7 and Theorem 2 to $\{v^r\}$ (the way Lemma 6 and Theorem 2 are applied to $\{y^r\}$) and then using Lemma 8 we finally get that $(\Lambda, x^r, \lambda) = (\Lambda, e^r, \lambda)$, and the proof is complete.

COROLLARY 1. *Let* λ, Λ, *and* $\{x^r\}$ *satisfy the hypothesis of Theorem* 3 *and assume that either*

(i) $\lambda \supseteq l^p$ *for some* $p > 1$, *or that*

(ii) λ *is averaging and repeating.*

If λ *is* Λ-*conullable in* Λ *under* $\{x^r\}$ *then*

$$(\Lambda, x^r, \lambda) = (\Lambda, w^r, \lambda) \neq (\Lambda, e^r, \lambda).$$

If λ *satisfies case* (ii) *and if* λ *is* Λ^*-*conullable in* Λ *under* $\{x^r\}$, *then*

$$(\Lambda^*, x^r, \lambda) = (\Lambda^*, w^r, \lambda) \neq (\Lambda^*, e^r, \lambda).$$

PROOF. Assume first that $\lambda \supseteq l^p$ for some $p > 1$. Since $e^r \to 0$ weakly in l^p, $e^r \to 0$ weakly in λ. (Indeed, $\lambda \supseteq l^p$ and so each $f \in \lambda'$ is also continuous on l^p [**4**, p. 203, Corollary 1].) The conclusion now follows easily from Theorem 3.

Assume next that λ is averaging and repeating. Then $V \in \Gamma_\lambda$ and $VA = I$, where $A = (a_{nk})$ is defined by the set of equations

$$a_{nk} = 1 \quad \text{for } k(k-1)/2 < n \leqq k(k+1)/2.$$

Since A is repeating, A also belongs to Γ_λ; hence, neither $(\Lambda^*, e^r, \lambda)$ nor (Λ, e^r, λ) is an ideal and the conclusion follows once again from Theorem 3.

COROLLARY 2. *Let* λ, Λ, *and* $\{x^r\}$ *satisfy the hypothesis of Theorem* 3. *If* λ *is also averaging, repeating, and expansive then* λ *is neither* Λ-*conullable nor* Λ^*-*conullable in* Λ *under* $\{x^r\}$.

PROOF. Define $A'=(a'_{nk})$ and $A''=(a''_{nk})$ by setting $a'_{n,3n-2}=a''_{n,3n-1}=1$. Since λ is contractive, A' and A'' belong to $\Gamma_\lambda\subseteq\Lambda$. Hence, $A=A'-A''$ also belongs to Γ_λ. Now A has a right inverse in Λ. Indeed, define $B=(b_{nk})$ by $b_{3k-2,k}=1$. Then $B\in\Gamma_\lambda\subseteq\Lambda$ (because λ is expansive) and $AB=I$. Moreover, $Aw^{3r-2}=-e^r$ and $Aw^n=0$ if $n\neq 3r-2$ for some r. Thus, if λ were Λ-conullable (resp., Λ^*-conullable) in Λ under $\{x^r\}$, then (by part (ii) of Corollary 1) $(\Lambda, x^r, \lambda)=(\Lambda, w^r, \lambda)\neq(\Lambda, e^r, \lambda)$ (resp., $(\Lambda^*, x^r, \lambda)=(\Lambda^*, w^r, \lambda)\neq(\Lambda^*, e^r, \lambda)$), while (by Theorem 2) $(\Lambda, w^r, \lambda)=(\Lambda, Aw^r, \lambda)=(\Lambda, e^r, \lambda)$ (resp., $(\Lambda^*, w^r, \lambda)=(\Lambda^*, Aw^r, \lambda)=(\Lambda^*, e^r, \lambda)$). Since these two conclusions contradict each other the proof of the corollary is complete.

COROLLARY 3. *Let* $x^r\in m$ *for each* r *and let* λ *be contractive with* $\lambda\supseteq l$ *and* $e\in\lambda^\beta$. *If every contractive matrix has a right inverse in* Λ *then* λ *is neither* Λ-*conullable nor* Λ^*-*conullable in* Λ *under* $\{x^r\}$.

PROOF. Since $e\in\lambda^\beta$, $w^r\notin\lambda$ for each r. Suppose that λ is Λ-conullable (resp., Λ^*-conullable) in Λ under $\{x^r\}$. Then, by Theorem 3, $(\Lambda, x^r, \lambda)=(\Lambda, e^r, \lambda)$ (resp., $(\Lambda^*, x^r, \lambda)=(\Lambda^*, e^r, \lambda)$). Let $E=(e_{nk})$ be a matrix such that $e_{1k}=1$ for each k. Then $E\in(\Lambda, x^r, \lambda)$ (resp., $E\in(\Lambda^*, x^r, \lambda)$) because $e\in\lambda^\beta$ and $x_1^r\to 0$, as $r\to\infty$. Thus, $E\in(\Lambda, e^r, \lambda)$ (resp., $E\in(\Lambda^*, e^r, \lambda)$), which is absurd because $Ee^r=e^1$ for each r.

4. Applications to the special spaces.

PROPOSITION 1. *If* c *is* Λ-*conullable in* Λ *under* $\{x^r\}$ *then* $(\Lambda, x^r, c)=(\Lambda, w^r, c)$ *and* $\Lambda=\Gamma_c$. *Moreover,* c *is never* Λ^*-*conullable in any* Λ *under any* $\{x^r\}$.

PROOF. Since c is both contractive and repeating, each contractive matrix has a right inverse in $\Gamma_c\subseteq\Lambda$. Therefore, by part (i) of Corollary 1, if c is Λ-conullable in Λ under $\{x^r\}$ then $(\Lambda, x^r, c)=(\Lambda, w^r, c)$. But then, by part (iii) of Theorem 1, $\Lambda=\Gamma_c$ and so the proof of the first statement is complete.

To prove the second statement assume that c is Λ^*-conullable in some Λ under some $\{x^r\}$. Since c is averaging (V is a regular Toeplitz matrix), it follows from part (ii) of Corollary 1 that $(\Lambda^*, x^r, c) = (\Lambda^*, w^r, c)$. Using again part (iii) of Theorem 1 we then get that $(\Gamma_c^*, x^r, c) = (\Gamma_c^*, w^r, c)$. But this is impossible because (Γ_c^*, w^r, c) is not an ideal in Γ_c. Indeed, define $A = (a_{nk})$, $B = (b_{nk})$, and $C = (c_{nk})$ by the set of equations:

$$a_{nn} = -a_{n,n+1} = 1;$$

$$b_{nk} = (-1)^k/2^n \text{ for } \sum_{i=1}^{n-1} 2^i < k \leqq \sum_{i=1}^{n} 2^i;$$

$$c_{21} = -c_{22} = 2;$$

$$c_{2n,k} = -2 \text{ for } 1 + \sum_{i=1}^{k-3} 2^i < n \leqq 1 + \sum_{i=1}^{k-2} 2^i,$$

$$= +2 \text{ for } 1 + \sum_{i=1}^{k-2} 2^i < n \leqq 1 + \sum_{i=1}^{k-1} 2^i.$$

Then all three matrices belong to Γ_c, $A \notin (\Gamma_c^*, w^r, c)$, $B \in (\Gamma_c^*, w^r, c)$, and $BC = A$. This completes the proof.

Let θ denote the set of compact matrices in Γ_v.

Proposition 2. *If v is Λ-conullable (resp., Λ^*-conullable) in Λ under $\{x^r\}$, then $(\Lambda, x^r, v) = (\Lambda, w^r, v) = \theta$ (resp., $(\Lambda^*, x^r, v) = (\Lambda^*, w^r, v) = \theta$) and $\Lambda = \Gamma_v$.*

Proof. v, like c, is contractive, repeating, and averaging. Thus, as in the preceding proof, we may use Corollary 1 and part (iii) of Theorem 1 to conclude that $(\Lambda, x^r, v) = (\Lambda, w^r, v)$ (or that $(\Lambda^*, x^r, v) = (\Lambda^*, w^r, v)$ in case v is Λ^*-conullable) and that $\Lambda = \Gamma_v$. But Sember [2] has shown that $(\Gamma_v, w^r, v) = \theta$, and so the proof follows from the observation that $(\Gamma_v, w^r, v) \supseteq (\Gamma_v^*, w^r, v) \supseteq \theta$.

It has already been pointed out (in the remarks preceding Lemma 3) that l^p $(p \geqq 1)$, c_0, and γ are never Λ-conullable in Λ under any $\{x^r\}$. We now show that the same is true for Λ^*-conullity. For the sake of completeness, however, we include the statement about Λ-conullity each time.

Proposition 3. *Let λ be either c_0 or m. Then λ is neither Λ-conullable nor Λ^*-conullable in any Λ under any $\{x^r\}$.*

Proof. This follows immediately from Corollary 2.

Since l and γ are both contractive and expansive, each contractive matrix in Γ_l (resp., Γ_γ) has a right inverse in Γ_l (resp., Γ_γ) and so the next result is an immediate consequence of Corollary 3.

PROPOSITION 4. *Let* λ *be either* l *or* γ. *Then* λ *is neither* Γ_λ*-conullable nor* Γ_λ^**-conullable in* Γ_λ $(=B[\lambda])$ *under any* $\{x^r\}$.

PROPOSITION 5. *Let* λ *be any one of the* l^p *spaces,* $p>1$. *Then* λ *is neither* Γ_λ*-conullable nor* Γ_λ^**-conullable in* Γ_λ $(=B[\lambda])$ *under any* $\{x^r\}$.

PROOF. We need only prove that λ is not Γ_λ^*-conullable.

Fix $p>1$, let $\lambda=l^p$, and let q be conjugate to p, i.e. $1/p+1/q=1$. Define $A=(a_{nk})$ and $B=(b_{nk})$ by the set of equations:

$$a_{nk} = 1/2^n \text{ for } \sum_{i=1}^{n-1} 2^{[iq]} < k \leqq \sum_{i=1}^{n} 2^{[iq]};$$

$$b_{nk} = 1/2^{[n(q-1)]} \text{ for } \sum_{i=1}^{k-1} 2^{[iq]} < n \leqq \sum_{i=1}^{k} 2^{[iq]},$$

where $[x]$ denotes the smallest integer which is greater than or equal to x. Then a straightforward computation shows that A and B belong to Γ_λ and that $AB=I$, the identity matrix. Moreover, $A\in(\Gamma_\lambda^*, e^r, \lambda)$ and so $(\Gamma_\lambda^*, e^r, \lambda)$ cannot be a proper ideal in Γ_λ. Hence, λ cannot be Γ_λ^*-conullable in Γ_λ under $\{x^r\}$. Indeed, since λ is contractive and expansive, each contractive matrix has a right inverse in Γ_λ and so, by Theorem 3, if λ were Γ_λ^*-conullable in Γ_λ under some $\{x^r\}$ then $(\Gamma_\lambda^*, x^r, \lambda)=(\Gamma_\lambda^*, e^r, \lambda)$, which is not possible.

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