ON THE 1-1 SUM OF TWO BOREL SETS[1]

RICHARD B. DARST

ABSTRACT. It is shown that there exists a Borel subset B of the real line and a homeomorphism ϕ of the real line such that the set $\{x-\phi(x); x\in B\}$ is not a Borel set.

Since the range, $\phi(B)$, of ϕ is a Borel set, this result complements a recent result [1] of Paul Erdös and Arthur H. Stone who showed that there exist Borel subsets C, D of the real line whose sum $C+D = \{x+y; x\in C, y\in D\}$ is not Borel.

We begin a verification of our assertion by recalling that there exist CBV (continuous, bounded variation) functions h on $I=[0, 1]$ to I such that $\operatorname{card}\{y; \operatorname{card}[h^{-1}(y)] > \aleph_0\} > \aleph_0$.

For the reader's sake, Arthur H. Stone very kindly contributed the following direct and elementary construction of a suitable h. Define h first on the usual Cantor ternary set C by

$$h\left(\sum_{n=1}^{\infty} a_n\cdot 3^{-n}\right) = \sum_{n=1}^{\infty} a_{2n}\cdot 9^{-n}$$

(where each of $a_1, a_2, \cdots$ is 0 or 2), and extend h to I by making it linear on each complementary interval. An elementary calculation then shows

$$|h(x) - h(y)| \leqq 3|x-y| \quad \text{for all } x, y \in I,$$

so that h is CBV; and clearly $\operatorname{card}[h^{-1}(y)]=c$ for each of the c numbers y in $h(C)$. Hence [2], there exists a Borel set A such that $h(A)$ is not Borel. Recall that h can be represented as the restriction to I of the difference, h_1-h_2, of two homeomorphisms. Let $B=h_1(A)$ and let $\phi=h_2\cdot h_1^{-1}$.

REFERENCES

1. Paul Erdös and Arthur H. Stone, *On the sum of two Borel sets*, Notices Amer. Math. Soc. 16 (1969), 968–969. Abstract #69T-B175.

2. R. Purves, *Bimeasurable functions*, Fund. Math., **58** (1966), 149–157. MR **33** #7487.

PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907

Received by the editors October 20, 1969 and, in revised form, January 8, 1970.

AMS Subject Classifications. Primary 0440, 2810.

Key Words and Phrases. Borel set, homeomorphism, real numbers.

[1] This research was supported in part by the National Science Foundation under grant no. GP-9470.